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DYNAMIC ANALYSIS OF A BRIDGE SUPPORTED WITH MANY VERTICAL SUPPORTS UNDER MOVING LOAD

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Abstract. This study is devoted to the investigation of dynamic analysis of a bridge supported with many vertical supports under a moving load. Each vertical support is modelled as a linear spring and a linear damper. The analysis is based on Euler-Bernoulli beam theory. The present method utilises the concept of distributed moving load, spring force and damping force, and avoids the use of matching conditions. Expressing these forces in terms of the unknown function of the problem, it highly simplifies obtaining an exact solution. An important property of Dirac delta distribution function is utilised in order to reach the exact solution. Considering one and three vertical supports, the response of the supported bridge is plotted and compared to different values of parameters. In the case of an undamped bridge with no support, the results are compared with those of previous papers.

Keywords: bridge, beam, damped, dynamic, support, moving load.

1. Introduction

Dynamic response of uniform beams, rods and bridges under moving loads has received considerable attention within the framework of Euler–Bernoulli beam theory. Especially, dynamic behaviour of railways of infinite lengths under moving loads has been extensively investigated by means of integral transformations. However, these methods are not suitable for analysing beams of finite length under moving loads, and involve complex algebra. Analysis by methods for suspension bridges (Grigorjeva *et al.* 2006; Idnurm 2006) and vibration analysis of beams under travelling loads were made (Ayre *et al.* 1950; Fryta 1972; Green, Cebon 1994; Inglis 1934; Inman 1994) and many other papers whose bibliographical account will not be made here.

In case of a concentrated force moving with a constant velocity along the beam, neglecting damping forces, Timoshenko (1927) found a solution and gave an expression for the critical velocity. Stanisic and Hardin (1969) also studied the same problem for the simply supported beam carrying a moving mass.

Esmailzadeh and Ghorashi (1992; 1995) investigated the behaviour of a finite beam carrying moving point masses. They have also dealt with the vibration analysis of beams due to partially distributed masses moving with constant velocity. Garinei (2006) examined the vibrations of simple beam like modelled bridge under harmonic moving loads. The effects of combined loads transmitted both by a single axle (constant component + harmonic components) and by multiple axle subsystems equally spaced have been studied. Using modal analysis, the deflection of a beam subjected to an axial tensile force N and a moving vertical force P, has been determined by Dahlberg (2006). This solution was exploited in a study of the deflections and wave propagations that occur in the contact wire of a railway overhead a catenary system.

In order to reduce the amplitude of vibrating beam, vertical supports can be used. The behaviour of the system in this case is practically very important and needs to be analysed in detail. However, since the region of solution is divided into many parts. In case of several vertical supports and $4\times(z+1)$ (here z is the number of vertical supports) boundary and matching conditions are produced in the study, the analysis results in a determinant with large dimensions and complex eigen values. This situation requires the use of pocket programs such as Matlab, Mathematica etc. Since our aim here is to obtain explicit expressions for the dynamic response of the mechanical system, a rather different but easily applicable method will

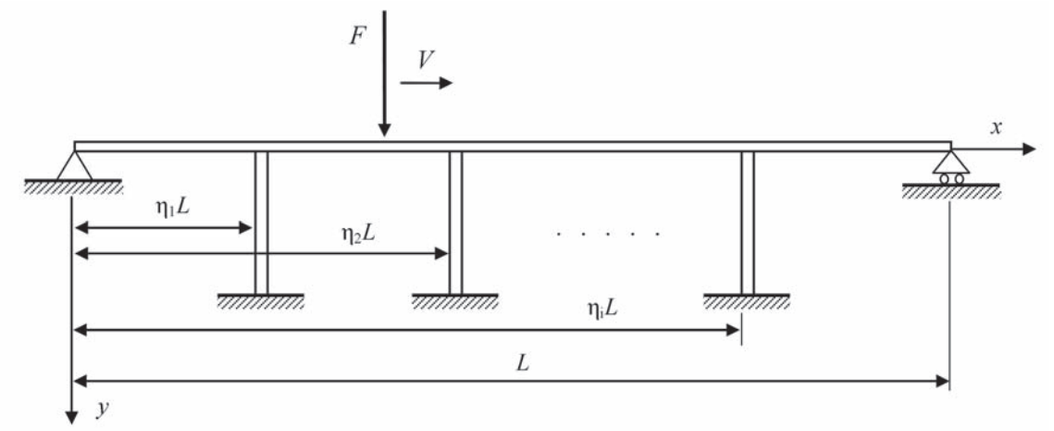

Fig 1. A bridge supported with many vertical supports under a moving load

be developed. As it is shown in the following part, this method avoids dealing with large matrices, determinants or their calculations, thus yielding formulae for everyday use without a computer. The second main difference of the method is that it is developed for beams (or bridges) of a finite length.

In this paper, as a different technique, the response of a simply supported finite bridge which is subjected to vertical supports and carries a moving force is investigated by transforming singular moving force, spring forces and damping forces into continuously distributed force fields by means of the property of Dirac delta distribution function. This enables us to study with eigenvalues of hinged-hinged finite beam.

2. Analysis

The system under consideration is shown in Fig 1. The uniform bridge is simply supported, bending rigidity *EI* and moving force *F* moves at a constant velocity V_0. Each support is modelled by a linear spring $(k_1, k_2, ..., k_i)$ and a linear damper $(c_1, c_2, ..., c_i)$ located at $\eta_i L$, as in Fig 1 and Fig 2 ($0 < \eta_i < 1$), where $i = 1, 2, \ldots, z$ (z – the number of support). Horizontal displacements of the bridge are neglected. The bridge is of a constant cross-section and a constant mass per unit length (m). The beam (bridge) damping is also inserted into the present analysis; it is proportional to the velocity of vibration in the form $F = c(\partial y/\partial t)$, where c is the damping coefficient.

According to the Euler-Bernoulli beam theory, the governing partial differential equation describing the transverse vibration of the bridge carrying the time-varying force $F(x,t)$ per unit length is (Stanisic, Hardin 1969):

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} = f(x,t), \tag{1}$$

where E – the modulus of elasticity, N/m^2; I – the second moment of area for the cross-section of the bridge, m^4; m – the mass per unit length of the bridge, kg/m; c – the damping coefficient of the bridge, N·s/m; y – the deflection of the bridge measured downwards from its equilibrium position when unloaded, m.

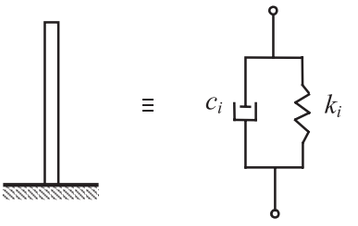

Fig 2. Physical modelling of vertical supports

In the present work, in contrast to the approach in (Gürgöze, 1997; Gürgöze, Mermertas 1998), we do not divide the domain of solution into parts. Instead, both damping force and the moving load are considered as distributed loads. Through this approach, matching conditions at $x_i = \eta_i L$ ($i = 1, 2, \ldots, z$) do not have to be used. As clarified later, the calculation of eigenvalues in this situation becomes very simple.

Let the transverse displacement of the bridge be expressed by $y(x,t)$. Thus the boundary conditions for the hinged-hinged beam (bridge) are:

$$x = 0, \quad y(0,t) = 0, \quad y''(0,t) = 0, \tag{2a}$$

$$x = L, \quad y(L,t) = 0, \quad y''(L,t) = 0. \tag{2b}$$

We assume that the displacement $y(x,t)$ in the forced vibration of the bridge has the form

$$y(x,t) = \sum_{n=1}^{\infty} a_n(t) X_n(x), \tag{3}$$

where $a_n(t)$ – unknown functions to be determined; $X_n(x)$ – the eigenfunctions of the same bridge in free vibration. In order to determine $X_n(x)$, we consider the free vibration of the system and write Eq (1) as:

$$\frac{\partial^4 \bar{\omega}}{\partial x^4} + b^2 \frac{\partial^2 \bar{\omega}}{\partial t^2} = 0\,, \quad b^2 = \frac{m}{EI}\,, \tag{4}$$

where $\bar{\omega}$ – the transverse displacement of the bridge in free vibration.

Using the method of separation of variables, and thus assuming $\bar{\omega} = X(x)T(t)$, Eq (4) can be separated into the following equations:

$$\frac{d^4 X}{dx^4} + \bar{k}^4 X = 0\,, \tag{5a}$$

$$\frac{d^2 T}{dt^2} - \frac{\bar{k}^4}{b^2} T = 0\,. \tag{5b}$$

The solution of Eq (5a) has the form:

$$X(x) = A\cos\bar{k}x + B\sin\bar{k}x + C\cosh\bar{k}x + D\sinh\bar{k}x\,. \tag{6}$$

Applying boundary conditions Eqs (2a) into Eq (6) gives $A = C = 0$. Other conditions (Eqs 2b) result in:

$$\begin{aligned} 0 &= B\sin\bar{k}L + D\sinh\bar{k}L\,, \\ 0 &= -B\bar{k}^2\sin\bar{k}L + D\bar{k}^2\sinh\bar{k}L\,. \end{aligned} \tag{7}$$

Thus after some simple calculations, nontrivial solution of eigenvalues is obtained:

$$\bar{k}_n = \frac{n\pi}{L}\,, \quad n = 1, 2, 3... \tag{8}$$

Since the bridge is hinged at both ends, eigenfunctions must be taken as:

$$X_n(x) = \sin\bar{k}_n x\,. \tag{9}$$

In the present case, substituting Eq (9) by Eq (3) yields:

$$y(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\bar{k}_n x\,. \tag{10}$$

Before we place Eq (10) into Eq (1), moving force *F*, spring forces $(Fy)_i$ and the viscous damping forces $(Fd)_i$ must be expanded into Fourier sinus series to avoid the matching conditions at $x_i = \eta_i L$, $i = 1, 2, ..., z$. It is a simple matter to show that the moving force is expressed as:

$$F = F(x,t) = F\delta(x - Vt) = \frac{2F}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi VT}{L}\sin\bar{k}_n x, \tag{11}$$

where $\delta(x - VT)$ is Dirac delta function.

Forces produced by each set of damper and spring can be expanded into Fourier sinus series as before. We start with the first group:

$$(F_d)_1 = c_1\dot{y}(\eta_1 L, t) = \sum(A_n)_1\sin\bar{k}_n x\,, \tag{12}$$

$$(F_y)_1 = k_1 y(\eta_1 L, t) = \sum(B_n)_1\sin\bar{k}_n x\,, \tag{13}$$

where

$$(A_n)_1 = \frac{2}{L}\int_0^L c_1\dot{y}(\eta_1 L, t)\sin\bar{k}_n x dx\,, \tag{14}$$

$$(B_n)_1 = \frac{2}{L}\int_0^L k_1 y(\eta_1 L, t)\sin\bar{k}_n x dx\,. \tag{15}$$

Substituting the expressions for $\dot{y}(\eta_1 L, t)$ into Eqs (14, 15) yields:

$$(A_n)_1 = \frac{2}{L}\int_0^L c_1\dot{y}(\eta_1 L, t)\sin\bar{k}_n x dx = \frac{2c_1\dot{a}_n(t)}{L}\sin\bar{k}_n(\eta_1 L), \tag{16}$$

$$(B_n)_1 = \frac{2}{L}\int_0^L k_1 y(\eta_1 L, t)\sin\bar{k}_n x dx = \frac{2k_1 a_1(t)}{L}\sin\bar{k}_n(\eta_1 L). \tag{17}$$

In the same way, the other groups can be written:

$$(F_d)_1 = \sum_{n=0}^{\infty}(A_n)_1\sin\bar{k}_n x = \frac{2c_2\dot{a}_n(t)}{L}\sin\bar{k}_n(\eta_2 L)\sin\bar{k}_n x, \tag{18}$$

$$(F_y)_2 = \sum_{n=0}^{\infty}(B_n)_2\sin\bar{k}_n x = \frac{2k_2 a_n(t)}{L}\sin\bar{k}_n(\eta_2 L)\sin\bar{k}_n x,$$

$$(F_d)_Z = \sum_{n=0}^{\infty}(A_n)_Z\sin\bar{k}_n x = \frac{2c_Z\dot{a}_n(t)}{L}\sin\bar{k}_n(\eta_Z L)\sin\bar{k}_n x, \tag{19}$$

$$(F_y)_Z = \sum_{n=0}^{\infty}(B_n)_Z\sin\bar{k}_n x = \frac{2k_Z a_n(t)}{L}\sin\bar{k}_n(\eta_Z L)\sin\bar{k}_n x.$$

We take care that the spring forces $(F_y)_i$ and damper forces $(F_d)_i$ are expressed in term of unknowns $a_n(t)$ and $\dot{a}_n(t)$'s respectively. Inserting Eqs (10), (11) and (19) into Eq (1) and arranging terms, we obtain:

$$\ddot{a}_n(t) + \left[\frac{c}{m} - \frac{2}{mL}\sum_{j=0}^{\infty}\left[c_j\sin\bar{k}_n(\eta_j L)\right]\right]\dot{a}_n(t) + \left[-\frac{EI}{m} - \frac{2}{Lm}\sum_{j=0}^{m}\left[k_j\sin k_j(\eta_j L)\right]\right]a_n = \frac{2F}{Lm}\sin\omega_F t. \tag{20}$$

We introduce the following abbreviations:

$$D_n = \frac{c}{m} - \frac{2}{mL}\sum_{j=0}^{\infty}\left[c_j\sin\bar{k}_n(\eta_j L)\right], \tag{21}$$

$$K = -\frac{EI}{m} - \frac{2}{mL}\sum_{j=0}^{m}\left[k_j\sin k_j(\eta_j L)\right], \tag{22}$$

$$F_0 = \frac{2F}{mL}\,, \quad \omega_F = \frac{n\pi V}{L}. \tag{23}$$

Now, Eq (20) reads:

$$\ddot{a}_n(t) + D_n\dot{a}_n(t) + Ka_n = F_0\sin\omega_F t. \tag{24}$$

The solution of homogenous part of Eq (24) is (Pala 2006):

$$(a_n)_h = b_1 e^{s_1 t} + b_2 e^{s_2 t}, \quad (25)$$

where b_1, b_2 – constants are to be determined. The roots s_1, s_2 – given by:

$$s_{1,2} = \frac{-D_n \pm \sqrt{D_n^2 - 4K}}{2}. \quad (26)$$

It is clear that three cases are valid. In examining these cases, it is convenient to define the modified critical damping coefficient, D_{ncr}, by:

$$D_{ncr} = 2\sqrt{K_n} = 2\omega_{mn}, \quad (27)$$

where ω_{mn} – the un-damped natural frequency corresponding to n mode. Furthermore, the non-dimensional number ς called the modified damping ratio, defined by:

$$\varsigma = \frac{D_n}{D_{ncr}}, \quad (28)$$

can be used to characterise the three types of solutions of the characteristic equation. Rewriting the roots given by Eq (26) yields:

$$s_{1,2} = \sqrt{K_n}\left[-\varsigma \pm \sqrt{\varsigma^2 - 1}\right]. \quad (29)$$

Modified damping ratio ς determines whether the roots are complex or real.

In case of modified damping ratio greater than $\varsigma > 1$, the discriminant is positive, resulting in a pair of distinct roots. The solution of the homogeneous part of Eq (24) then becomes

$$(a_n)_h = b_1 e^{s_1 t} + b_2 e^{s_2 t}, \quad s_1 < 0, \quad s_2 < 0, \quad (30)$$

which represents a non-oscillatory response. In Eq (30) b_1, b_2 are determined by the initial conditions.

In case of $0 < \varsigma < 1$, discriminant is negative, resulting in a complex conjugate pair of roots. These are:

$$s_{1,2} = \sqrt{K_n}\left[-\varsigma \pm \sqrt{1-\varsigma^2}\, i\right], \; i = \sqrt{-1}. \quad (31)$$

The solution of the homogeneous part of Eq (24) in this case is:

$$(a_n)_h = \bar{b}_1 e^{\alpha t}\cos(\beta t) + \bar{b}_2 e^{\alpha t}\sin(\beta t), \quad (32)$$

where $\bar{b}_1, \bar{b}_2$ – determined by the initial conditions.

Recall that this solution will die out in time due to the forcing terms F and F_d. Here:

$$\alpha = -\varsigma\sqrt{K_n}, \; \beta = \sqrt{K_n} \times \sqrt{1-\varsigma^2}. \quad (33)$$

In case of $\varsigma = 1$, which we call the modified critically damped case, the solution takes the form

$$(a_n)_h = \left(\hat{b}_1 + \hat{b}_2 t\right) \times e^{\alpha t}. \quad (34)$$

For a proper solution of Eq (24), let us assume:

$$(a_n)_p = \gamma_n \sin\omega_F t + \xi_n \cos\omega_F t. \quad (35)$$

Substitution of Eq (35) into Eq (24) gives:

$$\gamma_n = \frac{\left(K_n - \omega_F^2\right)F_0}{\left[\left(\omega_F^2 - K_n\right)^2 + D_n^2\omega_F^2\right]}, \quad (36)$$

$$\xi_n = \frac{-D_n\omega_F F_0}{\left[\left(\omega_F^2 - K_n\right)^2 + D_n^2\omega_F^2\right]}.$$

Combining Eqs (35) and (30) to have the general solution of Eq (24) for real s_1, s_2 gives:

$$a_n(t) = (a_n)_h + (a_n)_p = b_1 e^{s_1 t} + b_2 e^{s_2 t} + \gamma_n \sin\omega_F t + \cos\omega_F t. \quad (37)$$

For simplicity, we assume that the initial conditions are in the form:

$$y(x,0) = 0, \quad \dot{y}(x,0) = 0. \quad (38)$$

This assumption is not a restriction, and the initial conditions may be taken as non-zero when desired. Using these conditions in Eq (38), we can write:

$$a_n(0) = 0, \quad \dot{a}_n(0) = 0. \quad (39)$$

Applying these conditions to Eq (37), yields:

$$b_1 = \frac{\xi_n s_2 - \gamma_n\omega_F}{s_1 - s_2}, \quad (40)$$

$$b_2 = \frac{\gamma_n\omega_F - \xi_n s_1}{s_1 - s_2}.$$

If the roots s_1, s_2 – complex, then b_1, b_2 – given by:

$$b_1 = -\xi_n,$$

$$b_2 = \frac{\xi_n\alpha - \gamma_n\omega_F}{\beta}. \quad (41)$$

The problem is now completely solved.

3. Results and discussions

In order to prove the validity of the method, taking the values $EI = (2{,}07 \times 1{,}04) \times 10^5$ Nm^2, $c = 0$, $L = 10$ m, $F = 70 \times 9{,}81$ N, $m = 7 \times 9{,}81$ kg, without support ($k_i = 0$ and $c_i = 0$), given by Esmailzadeh and Ghorashi (1992) for the case of unsupported bridge, we have plotted in Fig 3. As clearly observed, the results of Esmailzadeh and

Ghorashi and the present work are in excellent agreement.

As a realistic numerical example, we assume that $c = 10^5$ N·s/m, $EI = 5 \times 10^5$ Nm^2, $m = 1000$ N/m, $L = 40$ m, $F_0 = 5 \times 10^5$ N, $V_0 = 20$ m/s. For the bridge supports ($k_1 = 3 \times 10^6$ N/m, $c_i = 2 \times 10^6$ N·s/m), we consider 3 cases: a) bridge with no support, b) bridge with unique support at the middle and c) bridge with equivalent supports at points $x_1 = 0{,}25L$, $x_2 = 0{,}5L$, $x_3 = 0{,}75L$.

Fig 4 compares the deflection y of the bridge at the point $x = 20$ m (midpoint) for values of ($k_1 = 3 \times 10^6$ N/m, $c_i = 2 \times 10^6$ N·s/m). Dotted line in Fig 4 corresponds to the bridge with no support. Dashed line in Fig 4 corresponds to the case of with unique support at the midpoint while solid line in Fig 4 represents the case of three equivalent supports located at $x_1 = 0{,}25L$, $x_2 = 0{,}5L$, $x_3 = 0{,}75L$. What is clearly seen is that the deflections decrease with the number of supports.

The variation of the displacement y versus x for instant $t = (0{,}4 - 0{,}8 - 1{,}2 - 1{,}6 - 2)$ s is shown in Figs 5a, 5b and 5c, respectively. As expected, it is clearly seen that the vibration amplitude of the bridge decreases with the number of supports.

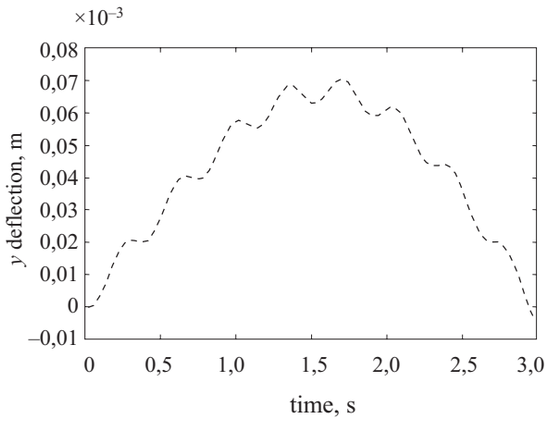

Fig 3. Variation of y with time t at the midpoint, $c = 0$, $L = 10$ m, $F = 70 \times 9{,}81$ N, $EI = (2{,}07 \times 1{,}04) \times 10^5$ Nm^2, (without support)

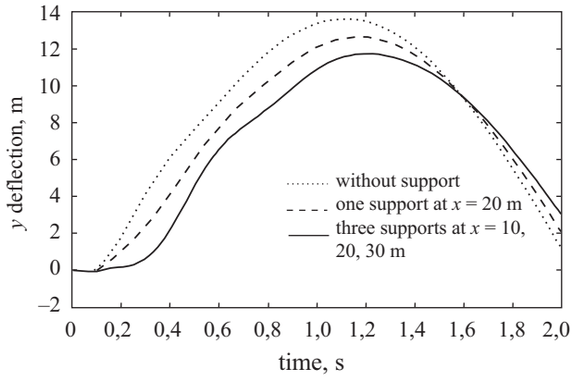

Fig 4. Variation of the displacement y versus time t at different points $x = 20$ m (midpoint)

Figs 6a and 6b compare the deflection y of the bridge at the midpoint for various locations of unique support. It is observed that the deflection amplitude of the bridge decrease when the support approaches the midpoint. Thus we conclude that the optimal solution for the location of the unique support is $x_1 = 0{,}5L$ at which the vibration amplitude becomes minimum.

An important problem arising in the analysis is that the behaviour of the bridge for large values of time t may deviate from the actual curve, the reason of which is the structure of Eq (24). Indeed, when the coefficient of $\dot{a}_n$ becomes too large, then the solution starts deviating from the actual curve. Therefore for large values of t, one must use the solution carefully. This point could be partially prevented by including damping property of bridge itself into Eq (1). This is the reason why Eq (1) in the present analysis involves the damping property of the bridge itself. On the other hand, time interval in the figures must not exceed the value of time necessary for the moving load to pass across the bridge. After this moment, the moving load starts again to be applied to the bridge for the second time.

The present method can be generalised to involve several moving loads. If, for example, we have N forces, thus they can be described as:

$$F_2 = \frac{2F_2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi V_2 (t - t_2)}{L} \sin \bar{k}_n x,$$

$$F_3 = \frac{2F_3}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi V_3 (t - t_2 - t_3)}{L} \sin \bar{k}_n x,$$

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$$F_N = \frac{2F_N}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi V_N (t - t_2 - t_3 - \ldots - t_N)}{L} \sin \bar{k}_n x.$$

All these terms take place on the right hand side of Eq (19) and do not cause any mathematical difficulty. Here, $t_2, t_3, \ldots, t_N$ show the differences among the durations of applied forces. $V_2, V_3, \ldots, V_N$ are the velocities of each force.

4. Conclusions

In this work, dynamic analysis of a bridge supported by many vertical supports under moving load has been performed using Fourier sinus series approach. In this way, a more realistic bridge model has been formed and analytically analysed. Depending on the constants involved in the problem, whether or not a damped bridge vibrates can be

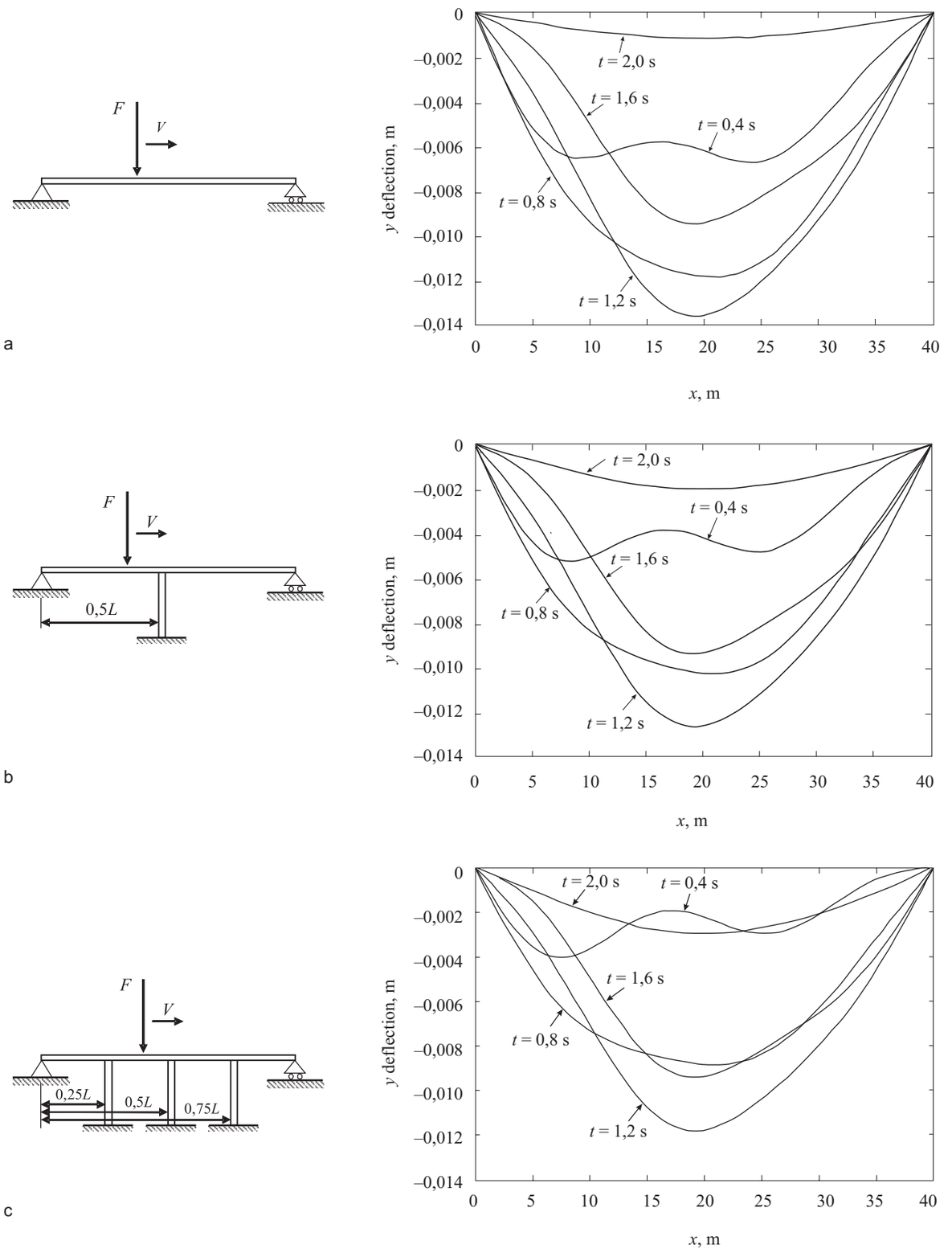
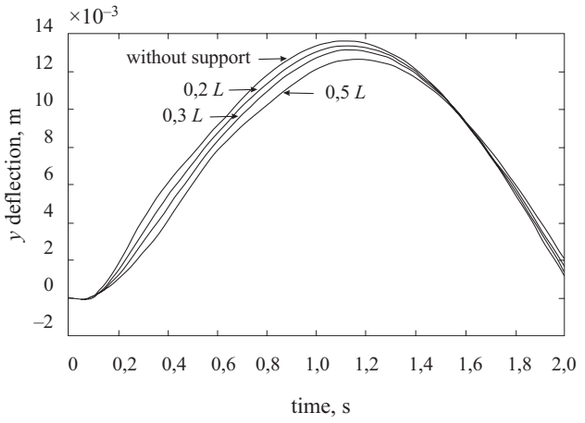
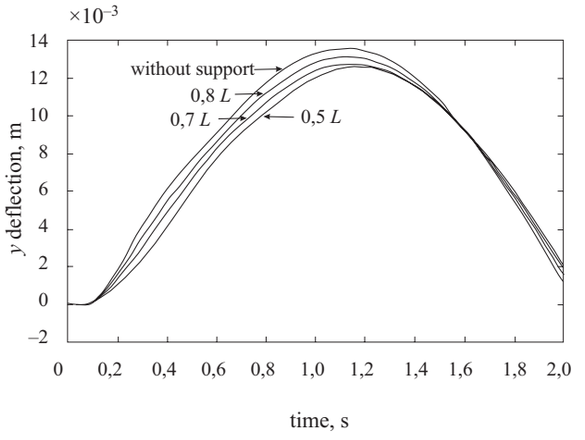


Fig 5. Variation of the displacement y versus x for instant $t = (0{,}4 - 0{,}8 - 1{,}2 - 1{,}6 - 2)$ s: a – bridge with no support, b – bridge with unique support located at $x = 20$ m ($k_1 = 3 \times 10^6$ N/m, $c_i = 2 \times 10^6$ N·s/m), c – bridge with three equivalent supports located at $x_1 = 10$ m, $x_2 = 10$ m, $x_3 = 10$ m ($k_1 = 3 \times 10^6$ N/m, $c_i = 2 \times 10^6$ N·s/m)



a



b

Fig 6. Variation of the displacement y versus time t for various locations of unique support: a – bridge with unique support between $0 < x_1 < 0{,}5L$; b – bridge with unique support between $0{,}5L < x_1 < L$

explained in a systematically way. In case of a bridge with no support, the results have been exemplified, and deflection curves have been obtained for some values of parameters involved in the problem. Calculations have revealed that the contributions made by the first three modes are dominant in the deflection of the bridge. Since the results explicitly depend on the number of modes, the present formulation best explains their effects on the solution.

The results of the method have been compared with those of Esmailzadeh and Ghorashi for the case of undamped beam (bridge) with no support, and found that they are in great agreement. However, the present method does not require the solution of coupled approximate dynamical equations as the work of Esmailzadeh and Ghorashi does. The present method seems easy especially, when several moving loads and supports are considered, while the other methods would require the numerical solution of coupled differential and algebraic equations. The method is even easier since no matching condition is utilised.

According to the present method, matching conditions do not constitute a problem in the solution, and frequency analysis is readily carried out.

As an extension of the method, one can also investigate the effect of curvature on the response of the bridge without making much change since the curved shape can be easily expanded into Fourier sinus or cosines series. When other methods are used, one cannot readily obtain analytical results and is forced into using numerical techniques.

References

Ayre, R. S.; Ford, G.; Jacobsen, L. S. 1950. Transverse vibration of a two-span beam under the action of a moving constant force, *Journal of Applied Mechanics* 17: 1–12.

Dahlberg, T. 2006. Moving force on an axially loaded beam–with applications to a railway overhead contact wire, *Vehicle System Dynamics* 44(8): 631–644.

Esmailzaden, E.; Ghorashi, M. 1992. *Beams carrying uniform partially distributed moving masses*. Technical report of Mechanical Engineering Dept, Sharif University of Technology, Tehran.

Esmailzaden, E.; Ghorashi, M. 1995. Vibration analysis of beams traversed by uniform partially distributed moving, *Journal of Sound and Vibration* 184(1): 9–17.

Fryba, L. 1972. *Vibration of solid and structures under moving loads*. Thomas Telford House, London, 13–33.

Garinei, A. 2006. Vibrations of simple beam-like modeled bridge under harmonic moving loads, *International Journal of Engineering Science* 44: 778–787.

Green, M. F.; Cebon, D. 1994. Dynamic response of highway bridges to very heavy vehicle loads. Theory and experimental validation, *Journal of Sound and Vibration* 170(1): 51–78.

Grigorjeva, T.; Juozapaitis, A.; Kamaitis, Z. 2006. Simplified engineering method of suspension bridges with rigid cables under action of symmetrical and asymmetrical loads, *The Baltic Journal of Road and Bridge Engineering* 1(1): 11–20.

Gürgöze, M. 1997. On the eigenvalues of viscously damped beams, carrying heavy masses and restrained by linear and torsional springs, *Journal of Sound and Vibration* 208(1): 153–158.

Gürgöze, M.; Mermertas, V. 1998. On the eigenvalues of a viscously damped cantilever carrying a tip mass, *Journal of Sound and Vibration* 216(2): 309–314.

Idnurm, J. 2006. Discrete analysis method for suspension bridges, *The Baltic Journal of Road and Bridge Engineering* 1(2): 115–119.

Inglis, C. E. 1934. A mathematical treatise on vibration in railway bridges, *Cambridge University Press*, Cambridge, 1–50.

Inman, D. J. 1994. *Engineering vibration*. Prentice Hall Inc., A Simon & Schuster Company, Englewoods Cliffs, New Jersey, 329–340.

Pala, Y. 2006. *Modern Uygulamalż Diferansiyel Denklemler* [Modern applied differential equations]. Nobel Publishing, Bursa, 533–536.

Stanisic, M. M.; Hardin, J. C. 1969. On response of beams to an arbitrary number of moving masses, *Journal of the Franklin Institute* 287: 115–123.

Timoshenko, S. 1927. Vibration of Bridges, *Transactions of the American Society of Mechanical Engineers* 53: 53–61.

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