



## DYNAMIC RESPONSE OF A SLIGHTLY CURVED BRIDGES UNDER MOVING MASS LOADS

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**Abstract.** In this research, the dynamic response of a slightly curved bridges under moving mass load is studied using an analytical approach. A solution method similar to the method of successive approximation has been used. The method has been exemplified for the special values of the variables. The effects of some variables have been specifically investigated. The results reveal that the inertial, centripetal and Coriolis forces must be involved in the analysis especially when the slightly curved bridges under moving loads with high speed are examined. Depending on the convexity and concavity of the initial curve, the effects of these forces become different. In a curved bridge, the moving mass affects the bridge more than that in a straight bridge with increasing the velocity of the moving mass. It has been observed that the forced vibration of the bridge is strongly influenced by the velocity of the moving mass. Many figures have been plotted to show clearly the effects of the variables.

**Keywords:** bridge, beam, curved, moving mass, response.

### 1. Introduction

The dynamic response of bridges under moving loads has been investigated for more than a century. The dynamic behavior of bridges has gained widespread attention with the influence of the railway transport. The pioneering studies in this area have started studying the effect of a constant force moving at uniform velocity on a straight beam. Dynamic stresses in a simply supported beam, under moving constant force were first solved by Kriloff (1905), then Timoshenko (1927). Interesting analyses were also presented by Idnurm (2006) and Grigorjeva *et al.* (2006). Hillerborg (1948) used Fourier's analysis and Biggs *et al.* (1959) the Inglis's technique (Inglis 1934). The problem is reviewed in detail by Timoshenko (1911) and Kolousek (1956a, b, 1967).

Stanišić and Hardin (1969) determined the dynamic behavior of a simply supported beam carrying a moving mass, which is interesting enough, but their method is not easily applicable to different boundary conditions.

Esmailzadeh and Ghorashi (1992) investigated the behavior of a beam carrying moving point mass, in which the inertial effect of the mass is considered. In a later paper, Esmailzadeh and Ghorashi (1995) have studied the transverse vibration of simply supported beams under moving mass load. The uniform mass load was assumed to be partially distributed on the beam.

The dynamic response of simple frames subjected to loads has been studied by Karaolides and Kounadis (1983),

Fertis (1987). Reis *et al.* (2008) investigated the dynamic analysis of a bridge supported with many feet under moving load. Inclusion of supports into the analysis divides the region of solution into many parts and brings about many conditions of continuity to be satisfied by the solution. Therefore, using Dirac's delta distribution functions, the singular forces must be written as equivalent distributed ones. This procedure allows us to solve the present problem together with the inclusion of inertial effect of the moving mass analytically.

In all the above studies, standard dynamic analyses were performed and the effects of centripetal and Coriolis forces associated with the mass of the moving load, transverse motion of the flexurally vibrating system were neglected. Neglecting the effects of the rotatory inertia of the mass, Michaltsos and Kounadis (2001) took these effects into account for a light unsupported straight bridge under a moving heavy load, and have shown that important differences in the vibration of the beam might occur. Matsagar and Jangid (2005) studied the viscoelastic damper connected to adjacent structures involving seismic isolation.

Reis *et al.* (2008) also investigated dynamic analysis of finite damped beams of small curvature under a moving constant force. This study is a more realistic model of this study. The moving load is not a constant force in reality. In this case, the effects of centripetal forces and Coriolis forces, curvature become much more important for the behavior of the bridge and must be inserted into the analysis. The present study is devoted to these effects.

## 2. Analysis

The system under consideration is shown in Fig. 1. In the analysis, the following assumptions are adopted:

- Initial curvature and damping property of the bridge are taken into account.
- Euler-Bernoulli beam theory is valid. Small deformations are considered. The bridge is of constant cross-section and constant mass per unit length (m).
- The force moves at a constant velocity  $v$ .
- The bridge is slightly curved. It is assumed that the initial curve can be expanded into Fourier series.
- The analysis is carried out for a simply supported beam.
- Horizontal displacements are neglected.
- Inertial, centrifugal and Coriolis effects of the moving mass  $M$  is inserted into the formulation.

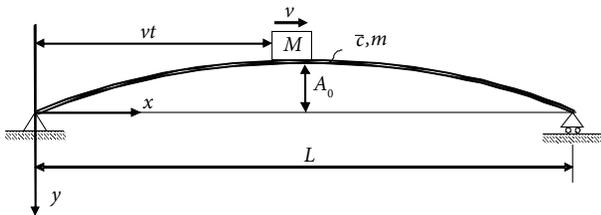


Fig. 1. Slightly curved bridge under a moving mass load

Under these assumptions, the governing partial differential equation takes the form

$$EI \frac{\partial^4 (y - y_0)}{\partial x^4} + m \frac{\partial^2 (y - y_0)}{\partial t^2} + \bar{c} \frac{\partial (y - y_0)}{\partial t} = M[g - a_m] \delta(x - a) \quad (1)$$

or

$$EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} + \bar{c} \frac{\partial y}{\partial t} = M[g - a_M] \delta(x - a) + EI \frac{\partial^4 y_0}{\partial x^4} \quad (2)$$

for small deformations. Here,  $\delta(x - a)$  is the Dirac's delta function. Since the transverse displacement  $y$  is a function of  $x$  and time  $t$ , the transverse acceleration  $a_M$  should be written as

$$a_M = \ddot{y} + v^2 y'' + 2v\dot{y}', \quad (3)$$

where  $\dot{y}' = \partial^2 y / \partial x \partial t$ ;  $y'' = \partial^2 y / \partial x^2$ ;  $\ddot{y} = \partial^2 y / \partial t^2$ . The second and third terms on the right side of Eq (3) correspond to the centrifugal and Coriolis accelerations. Inserting Eq (3) into Eq (2) yields

$$EIy^{iv} + m\ddot{y} + \bar{c}\dot{y}' = Mg\delta(x - a) - M[\ddot{y} + v^2 y'' + 2v\dot{y}']\delta(x - a) + Mv^2 y_0''\delta(x - a) + EIy_0^{iv} \quad (4)$$

or

$$EIy^{iv} + m\ddot{y} + \bar{c}\dot{y}' = EIy_0^{iv} + Mg\delta(x - a) - M[\ddot{y} + v^2 y'' + 2v\dot{y}']\delta(x - a) + Mv^2 y_0''\delta(x - a). \quad (5)$$

Boundary conditions of the hinged-hinged beam are given as

$$\begin{aligned} \underline{x=0} & & \underline{x=L} \\ y(0,t) &= 0, & y(L,t) &= 0, \\ y''(0,t) &= 0, & y''(L,t) &= 0. \end{aligned} \quad (6)$$

It is convenient to assume an expansion for the deflection  $y(x,t)$  in term of the combination of the linear free-oscillation modes,  $X_n = \sin(k_n x)$ ,  $n = 1, 2, \dots$ , which are those of hinged-hinged beam in this case. Thus,

$$y(x,t) = \sum_{n=1}^{\infty} a_n(t) X_n = \sum_{n=1}^{\infty} a_n(t) \sin(k_n x), \quad k_n = \frac{n\pi}{L}. \quad (7)$$

In the same manner, the initial shape of the bridge is also expanded into Fourier sinus series as

$$y_0(x) = \sum_{n=1}^{\infty} A_{n0} X_n = \sum_{n=1}^{\infty} A_{n0} \sin(k_n x), \quad k_n = \frac{n\pi}{L}. \quad (8)$$

In the present study, to exemplify the theory, let the initial curve of the form

$$y_0 = A_{n0} \sin\left(\frac{\pi x}{L}\right) \quad (9)$$

be assumed. Thus,  $A_{n0}$ 's read as

$$A_{n0} = A_0, \quad n=1, \quad A_{n0} = 0, \quad n=2,3,\dots \quad (10)$$

Recall that the boundary conditions, given by Eq (6), are automatically satisfied by the forms of Eqs (7) and (8).

Substituting Eq (7) and Eq (8) into Eq (5) yields

$$\begin{aligned} EI k_n^4 \sum_{n=1}^{\infty} a_n X_n + m \sum_{n=1}^{\infty} \ddot{a}_n X_n + \bar{c} \sum_{n=1}^{\infty} \dot{a}_n X_n = \\ EI k_n^4 \sum_{n=1}^{\infty} A_{n0} X_n + Mg\delta(x - a) - \\ M \left[ \sum_{n=1}^{\infty} \ddot{a}_n X_n - v^2 \sum_{n=1}^{\infty} a_n k_n^2 X_n + 2v \sum_{n=1}^{\infty} \dot{a}_n X_n \right] \delta(x - a) - \\ \left( Mv^2 \sum_{n=1}^{\infty} A_{n0} k_n^2 X_n \right) \delta(x - a). \end{aligned} \quad (11)$$

Multiplying both sides of Eq (11) by  $X_k$  ( $k \neq n$ ) and integrating from 0 to  $L$  lead to

$$\begin{aligned} \ddot{a}_n + \frac{\bar{c}}{m} \dot{a}_n + \omega_n^2 a_n = \omega_n^2 A_{n0} + \frac{2Mg}{mL} X_n(a) - \\ \frac{2M}{mL} X_n(a) \left[ \sum_{k=1}^{\infty} \ddot{a}_k X_k(a) - v^2 \sum_{k=1}^{\infty} k_k^2 a_k X_k(a) + \right. \\ \left. 2v \sum_{k=1}^{\infty} \dot{a}_k X_k'(a) \right] - \frac{2Mv^2}{mL} X_n(a) \left[ \sum_{k=1}^{\infty} A_{k0} k_k^2 X_k(a) \right]. \end{aligned} \quad (12)$$

Here,  $X_n(a)$  is the value of  $X_n(x)$  at the point  $x = a$ . The frequency  $\omega_n$ ,  $a$ , is given by

$$\omega_n^2 = \frac{EI k_n^4}{m}, \quad a = k_n vt. \quad (13)$$

In order to solve Eq (12), a technique developed by Kounadis (1992) will be used. This method has also been used in a different paper (Michaltsos, Kounadis 2001). In fact, this method is a different version of Picard's method applied to non-linear ordinary differential equations in which the non-linearity takes place on the right-hand side of the equation.

First, the homogenous part of Eq (12) must be solved, describing the damping ratio  $\zeta$  as

$$\zeta = \frac{\bar{c}}{2m\omega_n}. \quad (14)$$

The homogenous part of Eq (12) can be written in the form of

$$\ddot{a}_n + 2\zeta\omega_n\dot{a}_n + \omega_n^2 a_n = 0. \quad (15)$$

The characteristic Eq (15) has the form

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0, \quad (16)$$

from which one obtains

$$r_{1,2} = -\zeta\omega_n \mp \omega_n \sqrt{\zeta^2 - 1}. \quad (17)$$

It is clear that, depending upon value of  $\zeta$ , three cases are valid. In the case of damping ratio, greater than 1, the discriminant is positive, resulting in a pair of distinct roots. Thus, the solution of Eq (15) has the form

$$(a_n)_h = b_1 e^{r_1 t} + b_2 e^{r_2 t}, \quad r_1 < 0, \quad r_2 < 0, \quad (18)$$

which represents a non-oscillatory response.  $b_1, b_2$  are constant, yet to be determined. In the case, where  $0 < \zeta < 1$ , discriminant is negative, resulting in a complex conjugate pair of roots. The solution of the homogenous part is then given by

$$(a_n)_h = e^{-\zeta\omega_n t} [b_1 \cos \omega_d t + b_2 \sin \omega_d t], \quad 0 \leq \zeta < 1, \quad (19)$$

where  $\omega_d = \omega_n \sqrt{1 - \zeta^2}$  is the damped natural frequency in the unforced case. In the undamped case  $\zeta = 0$ , the transient solution is given by

$$(a_n)_h = b_1 \cos \omega_n t + b_2 \sin \omega_n t, \quad \zeta = 0. \quad (20)$$

Now, keeping only the first two terms on the right hand of the Eq (12), one obtains the solution of the non-homogeneous equation

$$\ddot{a}_n + 2\zeta\omega_n\dot{a}_n + \omega_n^2 a_n = \omega_n^2 A_{n0} + \frac{2Mg}{mL} \sin \omega_f t. \quad (21)$$

The proper solution for Eq (21) can be sought in the form

$$(a_n)_p = K_r + A_n \cos \omega_f t + B_n \sin \omega_f t. \quad (22)$$

Here,  $K_r, A_n$  and  $B_n$  are undetermined constants. Inserting this form into Eq (21), yields

$$\begin{aligned} K_r &= A_{n0}, \\ A_n &= -\left(\frac{2Mg}{mL}\right) \frac{2\omega_n \omega_f \zeta}{\left[(\omega_n^2 - \omega_f^2)^2 + 4\zeta^2 \omega_n^2 \omega_f^2\right]}, \\ B_n &= \left(\frac{2Mg}{mL}\right) \frac{(\omega_n^2 - \omega_f^2)}{\left[(\omega_n^2 - \omega_f^2)^2 + 4\zeta^2 \omega_n^2 \omega_f^2\right]}. \end{aligned} \quad (23)$$

Thus, the general solution to Eq (21) for  $0 < \zeta < 1$  takes the form

$$\begin{aligned} a_n &= (a_n)_h + (a_n)_p = \\ &e^{-\zeta\omega_n t} [b_1 \cos \omega_d t + b_2 \sin \omega_d t] + \\ &A_{n0} + A_n \cos \omega_f t + B_n \sin \omega_f t. \end{aligned} \quad (24)$$

Before replacing Eq (24) by Eq (12), recalling that  $X_n = \sin(k_n x)$ , Eq (12) is written as

$$\ddot{a}_n + 2\zeta\omega_n\dot{a}_n + \omega_n^2 a_n = F_n(t), \quad (25)$$

where the forcing term  $F_n(t)$  is given by

$$\begin{aligned} F_n(t) &= \omega_n^2 A_{n0} + \frac{2Mg}{mL} \sin(\omega_f t) - \\ &\frac{2M}{mL} \sin(\omega_f t) \left[ \sum_{k=1}^{\infty} \ddot{a}_k(t) \sin((\omega_f)_k t) - \right. \\ &\left. v^2 \sum_{k=1}^{\infty} k_k^2 a_k(t) \sin((\omega_f)_k t) + 2v \sum_{k=1}^{\infty} k_k \dot{a}_k(t) \cos((\omega_f)_k t) \right] - \\ &\frac{2Mv^2}{mL} \sin(\omega_f t) \left[ \sum_{k=1}^{\infty} A_{k0} k_k^2 \sin((\omega_f)_k t) \right]. \end{aligned} \quad (26)$$

Here

$$(\omega_f)_k = \frac{k\pi}{L} v, \quad k = 1, 2, \dots \quad (27)$$

$$\begin{aligned} A_{k0} &= 1, & n &= 1, \\ A_{k0} &= 0, & n &= 2, 3, \dots \end{aligned} \quad (28)$$

$$k_k = (k\pi)/L, \quad k = 1, 2, \dots \quad (29)$$

The solution to Eq (25) is expressed by Duhamel's integral

$$a_n(t) = \int_0^{\infty} F_n(\tau) h(t - \tau) d\tau, \quad (30)$$

provided that initial conditions are zero:  $a_n(t) = 0, \dot{a}_n(t) = 0$ ;  $h(t - \tau)$  is the response to Dirac's delta function  $\delta(t)$ , and is given by

$$h(t) = \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t. \quad (31)$$

Substituting the Eq (31) into (30) gives

$$a_n(t) = \frac{1}{\omega_d} \int_0^t e^{-\zeta\omega_n(t-\tau)} F_n(\tau) \sin[\omega_d(t-\tau)] d\tau. \quad (32)$$

Since the present problem involves non-zero initial conditions  $a_n(0) = A_{n0}$ ,  $\dot{a}_n(0) = 0$   $n = 1, 2, \dots$ , the solution of the homogenous part subjected to non-zero initial conditions into Eq (32) must be included. This solution is simply given by

$$a_n(t) = e^{-\zeta\omega_n t} [\bar{b}_1 \cos \omega_d t + \bar{b}_2 \sin \omega_d t], \quad (33)$$

where

$$\bar{b}_1 = A_{n0}, \quad \bar{b}_2 = \frac{\zeta\omega_n b_1}{\omega_d}. \quad (34)$$

As a result, the general solution of Eq (25), subjected to non-zero conditions, are obtained as

$$a_n(t) = e^{-\zeta\omega_n t} [\bar{b}_1 \cos \omega_d t + \bar{b}_2 \sin \omega_d t] + \frac{1}{\omega_d} \int_0^t e^{-\zeta\omega_n(t-\tau)} F_n(\tau) \sin[\omega_d(t-\tau)] d\tau. \quad (35)$$

### 3. Results and discussion

Eq (35) has been solved by means of a program written in Matlab for various values of variables. The first 4 terms for  $n$  and  $k$  have been taken in the calculations. Depending upon the number of  $n$  and  $k$ , the solution time may become too long. In order to see the overall picture, transversal deflection versus  $t$  (time) has been plotted for a specific example:  $L = 10$  m,  $b = 0.2$  m (width),  $h = 0.2$  m (height),  $m = 100$  kg/m,  $c = 1000$  Ns/m,  $E = 2.07 \cdot 10^{11}$  N/m<sup>2</sup>,  $A_0 = (-0,1-0,1)$  m (amplitude of the curvature, Fig. 2) and for the moving mass:  $M = 1000$  kg,  $v = (10-25-50)$  m/s.

To see the effects of curvature of the bridge and ve-

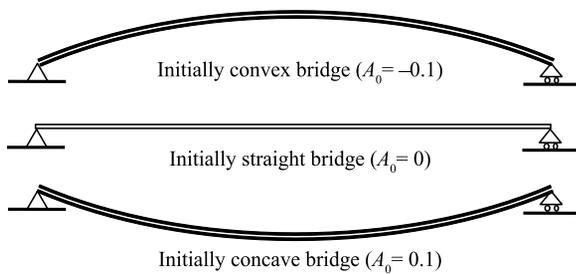


Fig. 2. Convex, straight and concave bridges

locity of the moving mass on the dynamic response of the bridge, the Figs 3, 4 have been plotted.

It is obvious, that in Fig. 3 initial curvature affects the response of the bridge. While a moving mass enters the bridge (if the bridge is concave), the mid-point of the bridge goes down faster; but if the bridge is convex, first the mid-point goes a little up for a very short time. It is interesting, when the time increases, the difference between the concave, straight and convex bridges decrease (Fig. 3). Another interesting result seen in Fig. 3, when the bridge is concave: the maximum deflection of mid-point is higher than with a straight and convex bridge. But at high velocities bigger deflections occur in convex bridges. So, velocity has an important effect on the response of the bridge. The effect of curvature decreases at lower velocity of the moving mass, as seen in Fig. 3a.

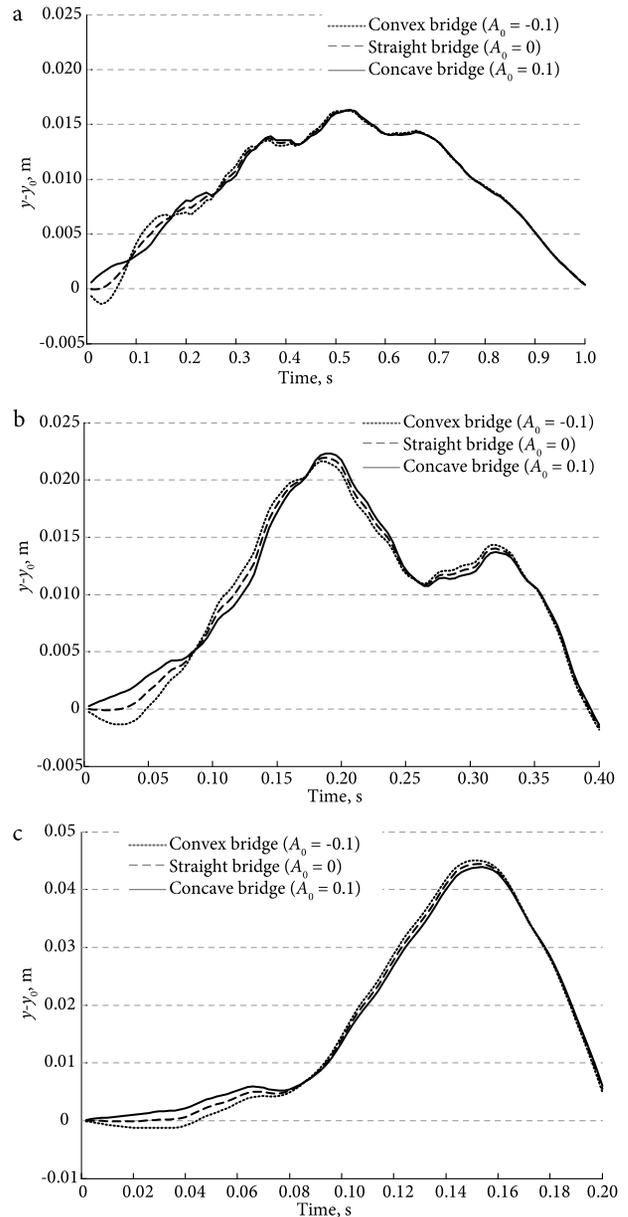
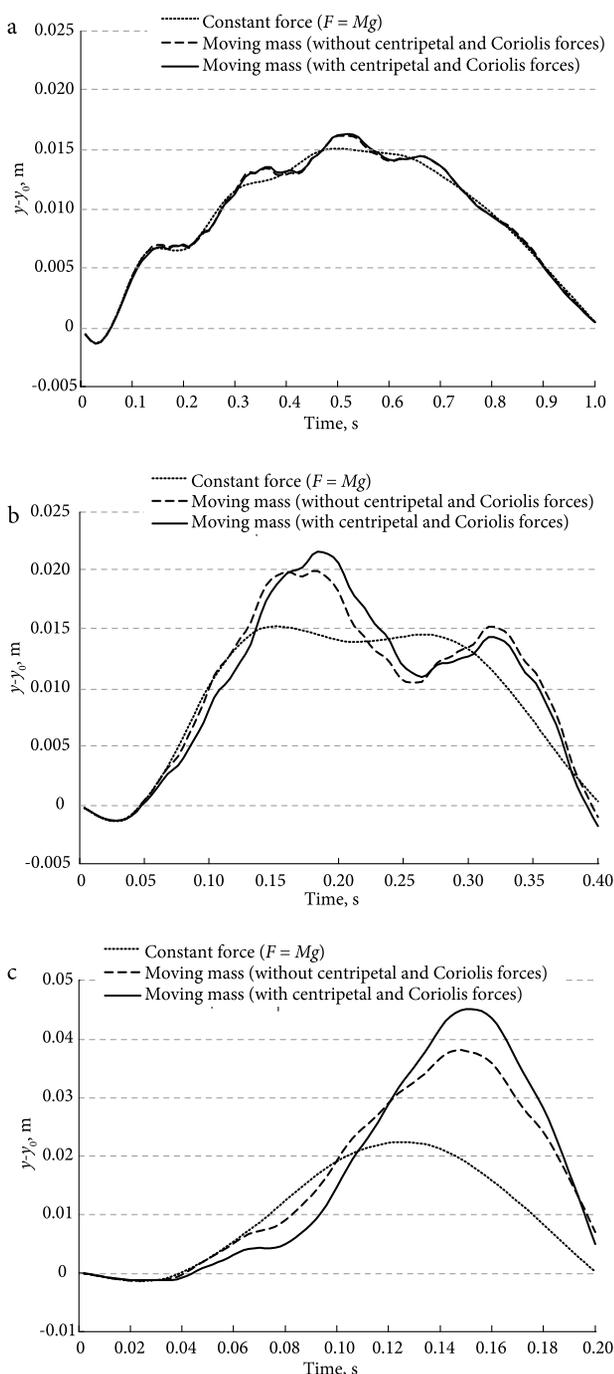


Fig. 3. Variation of the deflection at mid-point of the bridge under moving mass load (centripetal and Coriolis effects involved) with respect to time  $t$ : a -  $v = 10$  m/s; b -  $v = 25$  m/s; c -  $v = 50$  m/s

One of the main objectives of the present work is to show and compare the effects of constant force ( $Mg$ ), inertial forces, centripetal and Coriolis forces on the vibration of the beam in case of straight and slightly curved bridge (convex bridge). The deflection  $y-y_0$  as a function of time  $t$  at the mid-point ( $x = 5m$ ) is plotted in Figs 4a, 4b and 4c. It is clearly seen that as the velocity is increased, the effects of inertial, centripetal and Coriolis forces become more apparent (Figs 4b, 4c).



**Fig. 4.** Variation of the deflection at mid-point of the bridge with respect to time  $t$  ( $A_0 = -0.1$ , convex bridge): a –  $v = 10$  m/s; b –  $v = 25$  m/s; c –  $v = 50$  m/s

#### 4. Conclusions

In this study, the dynamic behavior of the slightly curved bridge under the moving mass loads have been investigated. The effects of inertial, centripetal and Coriolis forces together with the curvature have been taken into account. As also pointed by other writers, the problem under the present conditions is very difficult and requires specialized methods for an analytical solution. The solution has been obtained using the method proposed in Michaltsos, Kounadis 2001; Reis *et al.* 2008. This method is not a new one, but an extension of the method of successive approximations. While applying the method to the present problem, it has been noticed that it required too much time for increasing values of  $n$ . However, this might create a convergence problem. In order to obtain sensitive results, the number of steps in calculating the value of Duhamel’s integral must be increased in parallel to the increase in the value of  $n$ .

For the special values of the variables, the theory has been exemplified and the effects of the variables have been shown. The present calculations have revealed that the excessive vibration of the bridge can be controlled by intentionally giving a small curvature to the beam.

It has been observed in the calculations that the inertial forces, centripetal forces and Coriolis forces are very effective on the transverse vibration depending on the speed of the mass, and the curvature of the beam. Both terms must be involved in the analysis for the high-speed motion of the mass.

Without making much change in the present analysis, the method can be extended to the case of several moving mass loads. Using the present analysis, sprung mass model can also be analyzed. However, such an analysis would yield a system of coupled non-linear differential equations. When the forms of differential equations, especially when the type given by Eq (12) is considered, one should expect that the problem will be very difficult to analyze in case of a curved and damped bridge carrying sprung mass loads. In this case, purely numerical techniques rather than approximate analytical methods should be used, and the convergence of the solution must be carefully observed.

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