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EVALUATION OF STRUCTURAL INTEGRITY OF ASPHALT PAVEMENT SYSTEM FROM FWD TEST DATA CONSIDERING MODELING ERRORS

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Abstract. This study examines the structural integrity assessment technique used for the asphalt pavement system that considers the modeling errors introduced by material uncertainties. To this end, the artificial neural network is utilized to estimate the elastic modulus of soil layers by using the measured deflection data from the Falling Weight Deflectometer test. A wave analysis program for a multi-layered pavement system is developed based on the spectral element method for more accurate and faster calculation. The developed program is applied for the numerical simulation of the Falling Weight Deflectometer tests, specifically for the reliability analysis and the generation of training and testing patterns for the neural network. The effects of uncertainties in the material properties for modeling a given pavement system such as Poisson ratio and layer thickness are intensively investigated using the Monte Carlo Simulation. Results reveal that the amplitude of impact loads is most significant, followed by the layer thickness and the Poisson ratio, which are more significant on the max deflections than other parameters. The evaluation capability of the neural network is also investigated when the input data is corrupted by the modeling errors. It is found that the estimation results can be significantly deviated due to the modeling errors. To reduce the effect of the modeling error, (to improve the robustness of the algorithm), we proposed an alternative scheme in order to generate the training patterns taking into consideration any modeling errors. The study then concludes that the estimation results can be improved by using the proposed training patterns from an extensive numerical simulation study.

Keywords: FWD (Falling Weight Deflectometer), asphalt concrete (AC) pavement, neural network (NN), noise injection training, nondestructive structural integrity.

1. Introduction

The FWD (Falling Weight Deflectometer) test is operated by dropping a certain level of mass and measuring the max surface deflections using several sensors by equal spaces (Fig. 1). The FWD is most widely used to evaluate the structural integrity of an asphalt concrete (AC) pavement system. Many studies have been carried out with the objective of improving the testing method and the evaluation schemes of the conventional FWD test (Aavik *et al.* 2006; Al-Khoury *et al.* 2001a, 2001b; Bertulienė, Laurinavičius 2008; Choi *et al.* 2002; Dong *et al.* 2002; Jo *et al.* 2003; Kim, Kim 1998; Liang, Zhu 1995; Yun *et al.* 1995).

Most commercialized FWD systems utilize regression analysis that uses a large-scale database and/or the iterative optimization schemes for estimating the elastic modulus (E) of soil layers.

In this study, the artificial neural network (ANN) technique, which is one of the most widely used soft computing techniques in the civil engineering field, is applied to evaluate the structural integrity of a pavement system. In most cases, the modeling errors due to the uncertainties in the material properties such as layer thickness, Poisson ratio, unit weight, and damping ratio were not seriously considered while carrying out the inverse analysis using the FWD test data. However, it is almost impossible to exactly assign the material properties and it is therefore inevitable that a certain level of modeling errors will arise. With regard to this, it is very important to investigate the effects of modeling errors while considering the uncertainties in the material properties on the max deflection data of the FWD test. To this end, we carried out the reliability analysis that allowed for the uncertainties. An inverse analysis was also carried out that used the ANNs,

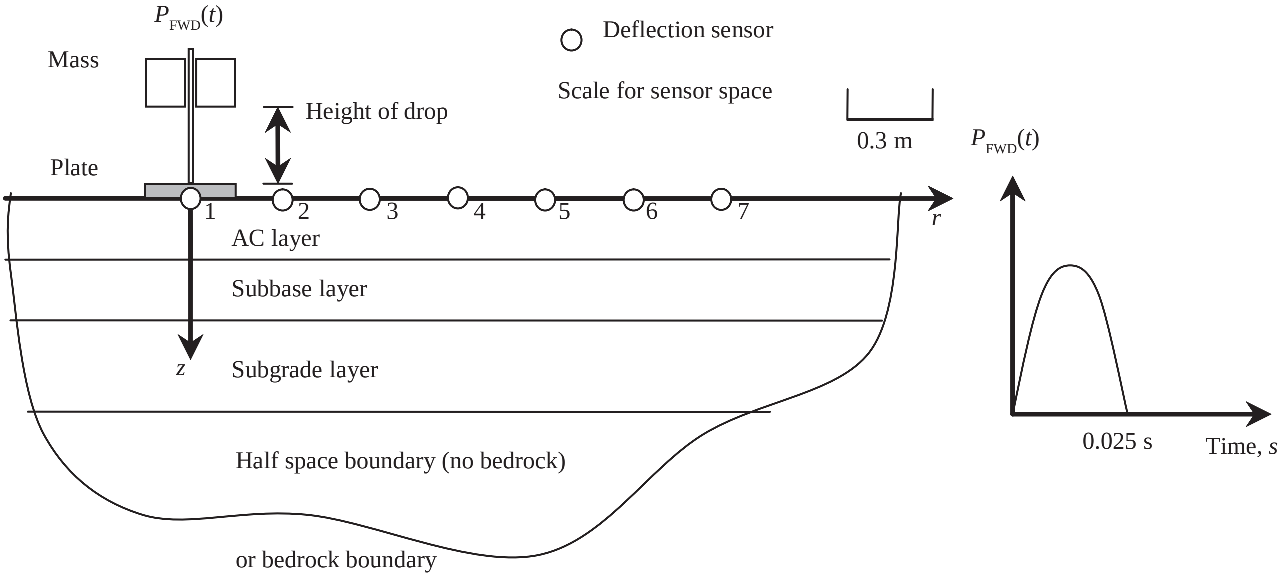

Fig. 1. Layered pavement structure and FWD test setup

both with and without modeling errors. We also proposed an alternative scheme in order to generate training patterns that consider the modeling error, utilizing the newly developed computer code, wave analysis of layered system (WALS), which is based on the spectral element method, for a more accurate and faster calculation. This was carried out in order to improve the estimation results in the cases where the modeling error occurred.

2. Theoretical backgrounds

2.1. Spectral element method

The wave propagation in a multi-layered system can be analyzed using several commercial computer codes such as ABAQUS and ANSYS. It can also be performed using many other specialized codes for wave propagation analysis such as BISAR, CHEVRON, ELSYM5, and WESLEA. Generally, too much computing time is required to generate a number of training patterns using general-purpose commercial codes. Furthermore, the analysis is carried out inaccurately when the specialized and compact-sized computer codes are applied. Therefore, a new computer code, WALS has been developed, which is theoretically based on the dynamic stiffness matrix method and the spectral element method to achieve a more accurate and faster calculation (Al-Khoury *et al.* 2001a; Kim, Mun 2008; Yun *et al.* 1995). The developed software has been used to simulate the FWD test, in order to carry out the reliability analysis and to generate the training and testing patterns for ANN modeling (Kim, Mun 2008).

2.2. Artificial neural networks (ANNs) and noise injection training

In this study, the *E* of pavement layers are estimated using a multi-layered perceptron neural network (NN), which is based on the max deflection data obtained from the FWD test as input data. An error back propagation algorithm is used as a training strategy to train the NNs, while the max deflection data and the corresponding *E* are used as input and output (target) data, respectively.

Since the theoretical backgrounds of the general training rules for NN's are referred to in many research papers, in this paper we intend to only introduce the basic concept of a noise injection learning algorithm (Matsuoka 1992; Yun, Bahng 2000). This algorithm improves the generalization capability of a NN by imposing random noise in the input data during the training process. This is carried out by a similar scheme to the proposed generation algorithm training patterns that consider modeling errors. A crucial problem with the BPNN (Back-propagation Neural Network) is its generalization capability. Usually, training patterns used for learning are taken from only a limited number of samples selected from a population of input and output patterns. Hence, a network successfully trained to a given set of samples may not provide the desired input and output associations for untrained patterns, particularly in the case where there are measurement noises and property uncertainties. Concerning this problem, several researchers have reported that adding a quantity of noise to the input patterns during the back propagation learning process can remarkably enhance the generalization capability of the resultant networks, if the mapping from the input space to the output space is smooth. Matsuoka (1992) found that the noise injected into the input reduces the sensitivity of the network to the variation of the input; that is, it creates smooth mapping from the input space to the output space (Matsuoka 1992; Yun, Bahng 2000). In the case of the FWD tests, the noise injection learning can be performed by imposing a certain level of random noise (α) to the input data as follows,

$$x_{ij} = f_j^{FWD}(p_i, h_i, E_i \rho_i, \gamma_i)(1+\alpha), \tag{1}$$

where x_{ij} – the input data (the max deflection at the jth measuring point of the ith training pattern); p_i, h_i, E_i, ρ_i

and γ_i – the amplitude of impact load, layer thickness, E, Poisson ratio, and damping ratio of the ith training pattern respectively; α – the Gaussian random noise with 0 mean and σ (standard deviation). Since we need to estimate only the E in this problem, the other parameters can be fixed as the representative values, i.e., $p_i = p$, $h_i = h$, $\rho_i = \rho$, $\gamma_i = \gamma$. The function f_j^{FWD} represents the max deflection at the j^{th} measuring point by the FWD test. By introducing the noise injection learning algorithm, similarly to Eq (1), to the NN, a more reliable estimation can be carried out when the measurement data is corrupted by measurement noise. However, the noise injection learning can only reduce the effects of the measurement noise and it is not sufficient to reduce the effects of the modeling errors included in the numerical model. Therefore, we proposed an alternative generation scheme of training patterns that considers the modeling errors, as follows:

$$x_{ij} = f_j^{FWD}\left(\begin{array}{l} p_i\left(1+\beta_p\right), h_i\left(1+\beta_h\right), E_i\left(1+\beta_E\right), \Rightarrow \\ \rho_i\left(1+\beta_\rho\right), \gamma_i\left(1+\beta_\gamma\right) \end{array}\right), \qquad (2)$$

where β_k – the Gaussian random noise with 0 mean; σ_k – the standard deviation for the k^{th} material parameter. It is understood that Eq (1) for the conventional noise injection learning takes into account the output errors in the FWD test.

Also, Eq (2) for the proposed generation scheme considers the input error in the numerical model, especially an inevitable heterogeneity of E of the pavement layer that was made during the compaction process.

3. Example analysis

3.1. Example asphalt pavement systems

Predicting the depth to the bedrock is one of the practical and yet most difficult issues that need to be addressed in the analysis of in situ pavement data. Two different layered pavement systems were therefore considered in order to create 2 networks according to the bedrock depth. 1 pavement system consists of 3 layers, i.e. AC surface, subbase, and half space subgrade layer (i.e. no bedrock), and the other system is composed of 4 layers, i.e. AC surface, subbase layer, subgrade layer and bedrock (Fig. 1).

By using these example systems, a reliability analysis is carried out in order to investigate the effects of modeling errors introduced by material uncertainties on the max deflections. Representative material properties are shown in Table 1.

Table 1. Modeling parameters for pavement systems

Parameters	AC surface	Subbase	Subgrade
Unit weight, kg/m^3	2350	2100	1900
Thickness, m	0.30	0.40	4.30
E, MPa	3500 (150–21 000)	350 (150–750)	100 (35–210)
Poisson ratio	0.35	0.40	0.45
Damping ratio	0.05	0.02	0.05

Furthermore, the range of the E is shown in the parenthesis, i.e., the ranges for the AC surface, subbase, and subgrade layers are in 150~21 000 MPa, 150~750 MPa, and 35~210 MPa, respectively. 4 types of structural integrity conditions are considered in order to investigate the evaluation capability of the NN as an inverse analysis tool. 4 conditions represent the healthy state for all layers (condition I), the poor surface layer case (condition II), the poor subbase layer case (condition III), and the poor subgrade layer case (condition IV) (Table 2).

Fig. 2 shows the max deflection curves for the different pavement conditions shown in Table 2. The least value of the max deflection at w_1 is about 0.2 mm for the case of condition I. The largest value of the max deflection is about 0.3 mm for the case of condition IV, which has a poor subgrade. However, the max deflection curves have similar trends, with the exception of w_1 in the cases of conditions II and III.

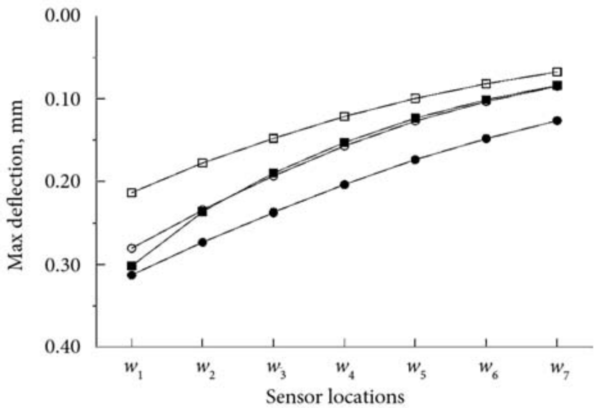

Fig. 2. Comparison of max deflection curves for conditions: —□— I; —■— II; —○—III; —●— IV

Table 2. E for integrity conditions

Condition	Elastic modulus (E), MPa			Description
	AC surface ($E1$)	Subbase ($E2$)	Subgrade ($E3$)	
I	10 000	500	150	Healthy condition
II	5000	450	120	Poor surface
III	8000	250	120	Poor subbase
IV	8000	450	50	Poor subgrade

3.2. Reliability analysis

For the inverse analysis, (the estimation of E of each pavement layer), the information on layer thickness, Poisson ratio, unit weight and damping ratio are required in order to model the behavior of the asphalt pavement system during the FWD test. The values of material properties can be assigned using conventionally acceptable values. For example, the Poisson ratio is usually considered in the range of 0.3–0.5 for the general pavement structures and the layer thickness can be decided by design drawings or a GPR test (Ghasemi, Abrishamian 2007; Loizos, Plati 2007). However, it is very difficult to exactly decide the material properties and there are inevitable uncertainties because each layer was constructed by compaction and rolling in the field. Therefore, there is a certain level of modeling error during an inverse analysis. In the case of forward analysis, the effects of the uncertainties can be investigated by carrying out a reliability analysis. Therefore, the effect of modeling errors due to uncertainty is generally not particularly significant. Errors in analysis results such as max deflections are not significantly corrupted by the modeling error and the results usually fall into the acceptable range in the view of an engineer's judgment. However, in the case of inverse analysis, the modeling error can lead to a significant estimation error and sometimes the inversion process can diverge. This is caused by modeling error since the modeling error is one of the ill-posedness in inverse analysis. Therefore it is very important to investigate the effect of the material uncertainties on the max deflection data, which is important information for inverse analysis that uses the FWD test. It is also important to develop an alternative scheme to reduce this ill-posedness due to modeling error.

In this study, the reliability analysis is first carried out to investigate the effects of the material uncertainties on the max deflections by utilizing the Monte Carlo Simulation for the above mentioned 4 type integrity conditions of 2 example pavement systems. Due to the lack of probabilistic information on the pavement system, we consider the distribution of material properties as the normal distribution. The values in Table 1 are regarded as mean values for layer thickness, unit weight, Poisson ratio and damping ratio. The coefficients of variations (*COV*'s) for all parameters are assumed to be 5% in this study. For each integrity condition, 100 samples are generated for the Monte Carlo Simulation. The *COV*'s of the max deflections are shown in Figs 3, 4. It can be found that the effect of the uncertainty in the amplitude of impact loading is most significant for almost all cases on the 2 systems. It can also be seen that the uncertainty in the layer thickness has a greater affect in the measuring points near the impact source, while the uncertainties in the E mainly affect the points that are far

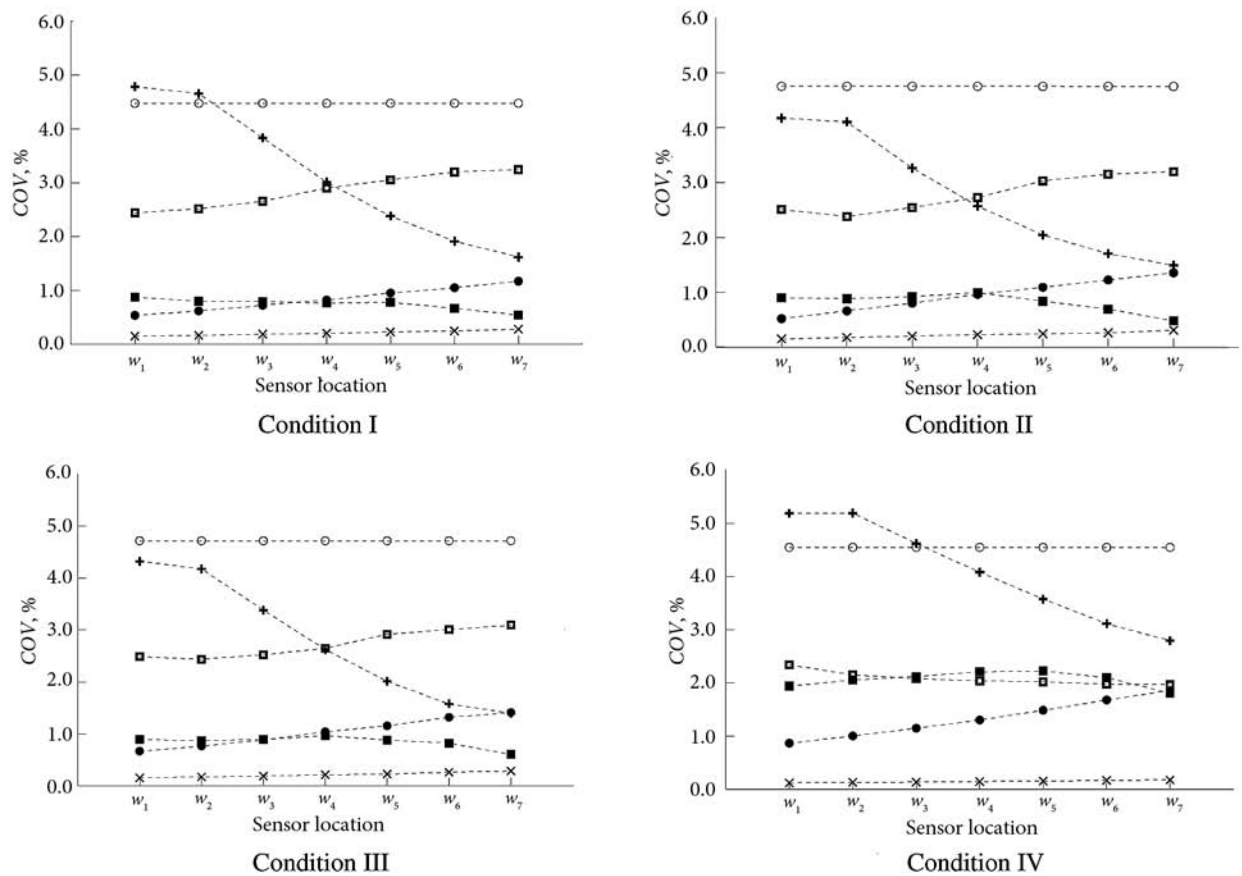

Fig. 3. *COV*'s of max deflection data according to material uncertainties for half space boundary case: --⊡-- – elastic modulus; --+-- – layer thickness; --⊖-- – force amplitude; --■-- – Poisson ratio; --●-- – density; --×-- – damping ratio

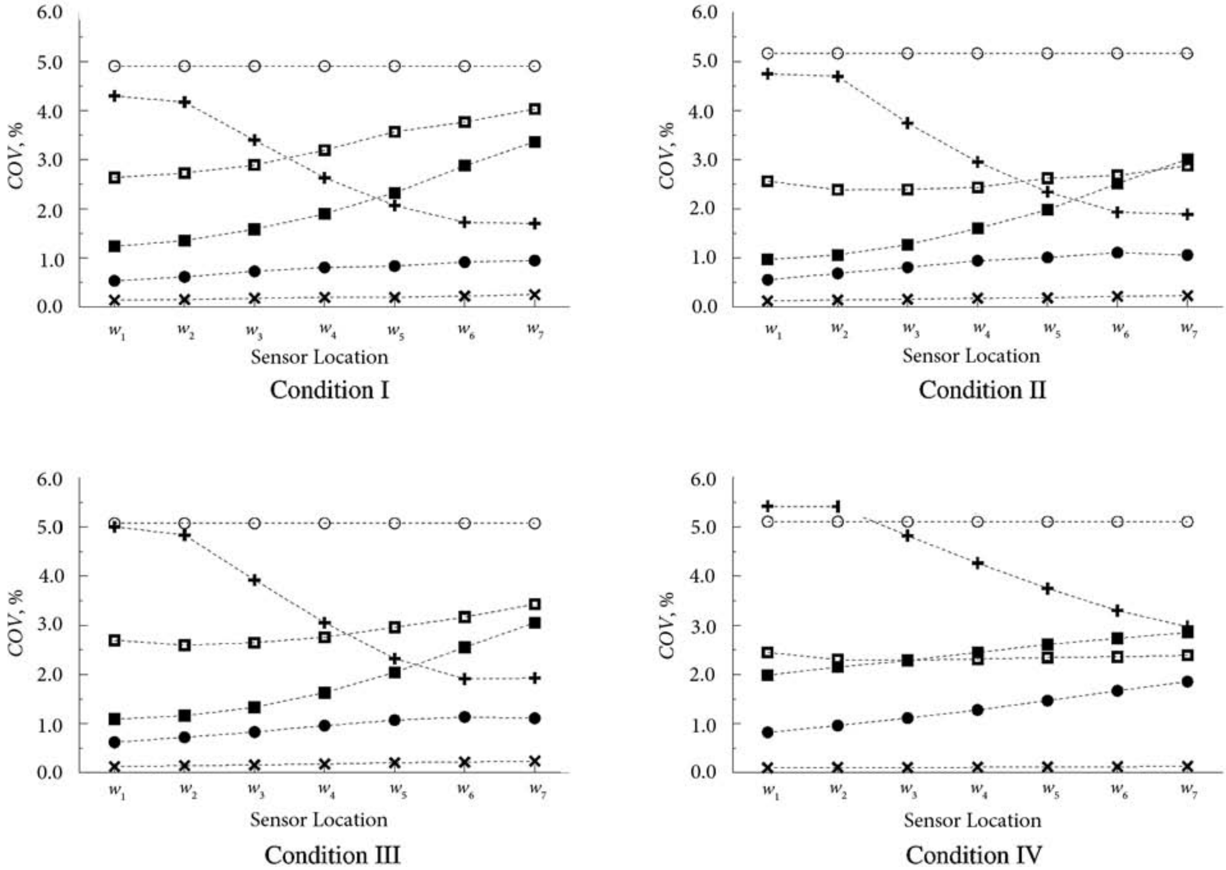

Fig. 4. *COV*'s of max deflection data according to material uncertainties for bedrock boundary case--□-- – elastic modulus; --+-- – layer thickness; --○-- – force amplitude; --■-- – Poisson ratio; --●-- – density; --×-- – damping ratio

from the impact source. The uncertainties in density, Poisson ratio and damping ratio are less significant. The damping ratio is not strongly related to the max deflection, since the damping ratio usually controls the decaying trend under impact loading.

3.3. Inverse analysis using Neural Networks (NNs)

The NN consists of 4 layers, i.e. an input layer, 2 hidden layers, and an output layer, while each layer consists of 7, 15, 10 and 3 neurons, respectively. The max deflection values for the 7 measuring points and the E for 3 pavement layers are utilized as the information for input layer and output layer. Therefore, each training pattern consists of 7 input values (max deflections) and the 3 target values (E). The min number of training patterns is determined as 1000, which is recommended by Yun and Bahng (2000), Vapnik and Chervonenkis (1971) to be about 2 times the number of total synaptic weights. The total learning epoch is fixed as 3000 iterations, the 1st – 1000 epoch with a constant learning rate of 0.1, the 2nd – 1000 epoch with a learning rate of 0.01 and the final – 1000 epoch with a learning rate of 0.001. Tables 3, 4 show the estimation results of E of each system for the structural integrity conditions I, II, III and IV, by using the numerical simulation data without modeling error. These results are obtained for the ideal cases (no modeling error exists), and it can be easily found that the estimation is very successfully carried out with very small estimation errors irrespective of the boundary condition (max 2.29% and 3.37% for AC surface layer, max 2.6% and 1.72% for subbase layer, and max 2.1% and 0.87% for the subgrade layer, respectively). It can be concluded that the NN can be a useful tool for inverse analysis of all pavement systems using the FWD test data when no modeling error exists.

Fig. 5 shows the comparison of the estimation results for the noise-injected testing patterns from the conventional NNs, which are used for obtaining the results in Tables 3, 4 and from the NNs that are trained using the noise injected training patterns, which are newly proposed to consider the material uncertainties (Eq (2)). It can be determined that the conventional networks cannot predict the E of the 2 pavement systems as precisely as they did for the ideal test patterns. In the case of the AC surface layer, the estimation errors are up to within the range of 8–11% for the half space boundary case and 7–10% for the bedrock boundary case obtained by using the conventional NN. Furthermore, the estimation error is reduced to within the range of 5–8% and 2–3% by adopting the proposed NN. In the cases of subbase and subgrade layers for the half space boundary case, the estimation errors are also reduced from 13–21% to

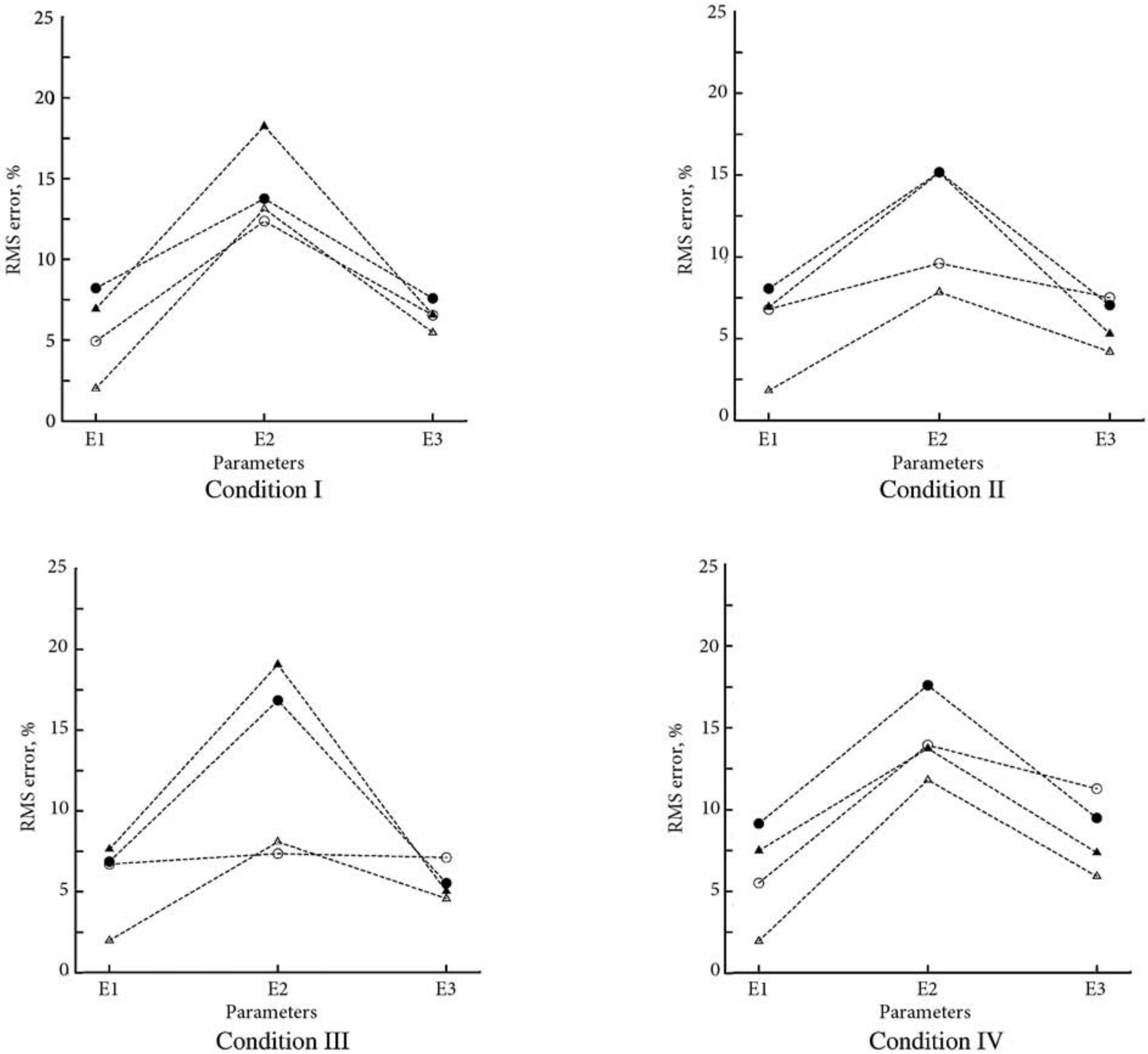

Fig. 5. Comparisons between conventional and proposed NN of estimation error corresponding to integrity condition --△-- – bedrock case: proposed NN; --▲-- – bedrock case: conventional NN; --○-- – no bedrock: proposed NN; --●-- – no bedrock: conventional NN

Table 3. Estimation results by NN without modeling error for half space boundary case

	Conditions	I	II	III	IV
AC surface	$E_{estimated}$, MPa	9995.8	4907.5	8125.5	7817.1
	E_{target}, MPa	10 000.00	5 000.00	8 000.00	8 000.00
	Error, %	0.042	1.85	–1.569	2.2862
Subbase	$E_{estimated}$, MPa	512.77	446.98	247.65	438.29
	E_{target}, MPa	500.00	450.00	250.00	450.00
	Error, %	–2.554	0.671	0.94	2.6022
Subgrade	$E_{estimated}$, MPa	148.79	117.48	118.66	50.36
	E_{target}, MPa	150.00	120.00	120.00	50.00
	Error, %	0.81	2.1	1.1167	–0.72

Table 4. Estimation results by NN without modeling error for bedrock boundary case

	Conditions	I	II	III	IV
AC surface	$E_{estimated}$, MPa	10 031.23	5168.52	7993.79	8073.26
	E_{target}, MPa	10 000.00	5000.00	8000.00	8000.00
	Error, %	–0.31	–3.37	0.08	–0.92
Subbase	$E_{estimated}$, MPa	508.61	443.47	254.30	456.15
	E_{target}, MPa	500.00	450.00	250.00	450.00
	Error, %	–1.72	1.45	–1.72	–1.37
Subgrade	$E_{estimated}$, MPa	150.30	119.31	119.87	50.43
	E_{target}, MPa	150.00	120.00	120.00	50.00
	Error, %	–0.20	0.58	0.11	–0.87

8–16%, and from 6–11% to 6–13% respectively. While in the cases of subbase and subgrade layers for the bedrock boundary case, the estimation errors are also similarly reduced from 15–23% to 7–14% and from 5–9% to 4–7% respectively. From the above results, it can be concluded that:

- the estimation error can be increased significantly when modeling error exists;
- the proposed generation scheme of noise-injected training patterns can effectively handle the modeling error caused by material uncertainties, irrespective of the pavement system boundary.

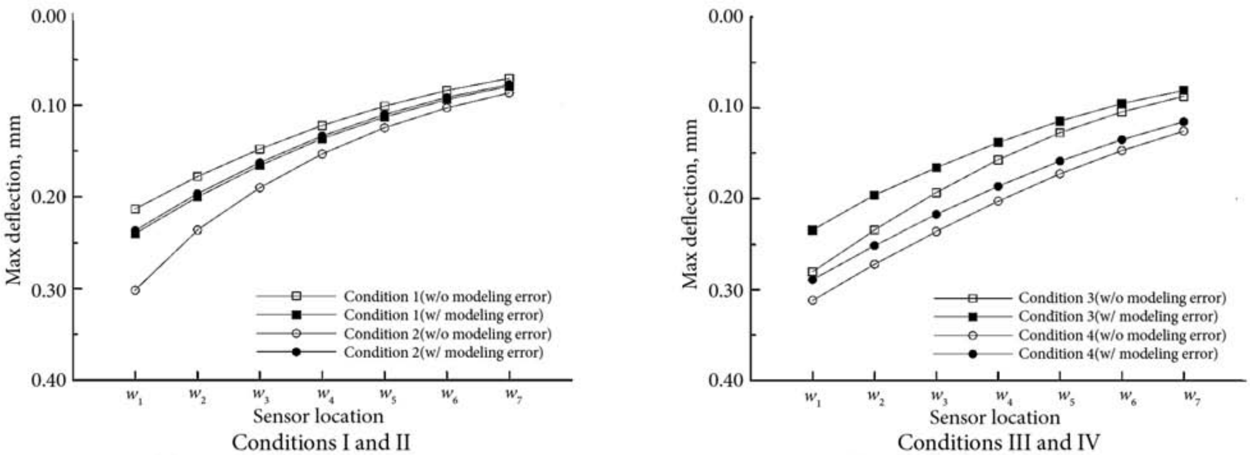

Fig. 6. Max deflection curves with and without modeling errors for integrity conditions for the half space boundary case

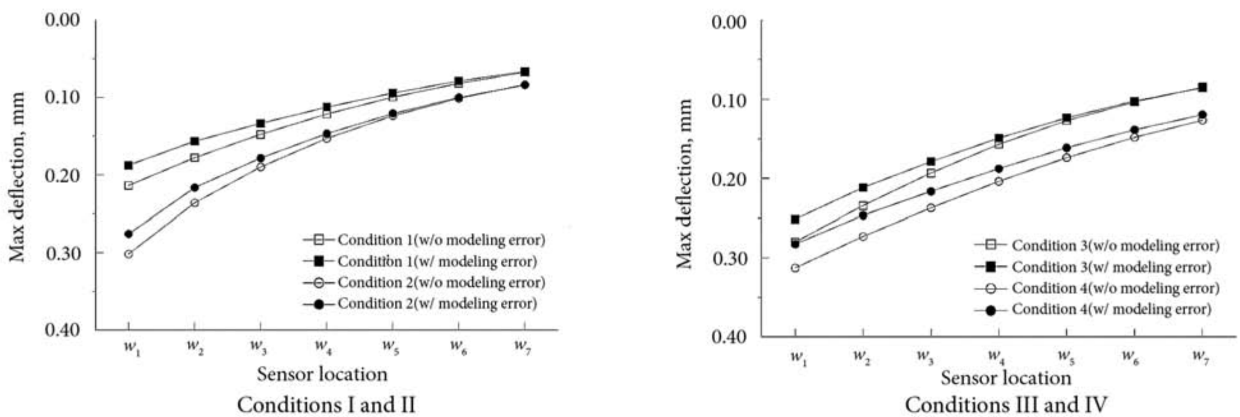

Fig. 7. Max deflection curves with and without modeling errors for integrity conditions for bedrock boundary case

We also compared the max deflection curves and the estimation results for the most severely corrupted cases by using a conventional network for all four damage conditions.

Fig. 6 shows the max deflection curves with and without modeling errors for the half space boundary corresponding to damage states I, II, III and IV. It can be observed that there are considerable discrepancies between 2 curves for the same integrity condition. Fig. 7 shows almost the same result for the bedrock boundary system. Fig. 8 shows the inverse analysis results obtained by the conventional and proposed NN for these typical cases shown in Fig. 6. It can be seen that the *E* can be overestimated up to 40% especially for the intermediate subbase layer, *E*2. However, the estimation errors can be reduced from 8–20% to below 3% for the AC surface layer and from 30–40% to below 8% for subbase layer.

Fig. 9 shows the inverse analysis results obtained by the conventional and proposed NN for the deflection curves of the bedrock boundary shown in Fig. 7. It can also be seen that the *E* can be overestimated even up to 80%. However, the estimation errors are reduced from 5–20% to below 5% for the AC surface layer and from 40–80% to 5–20% for the subbase layer by introducing the generation scheme of training patterns that account for the uncertainties in material properties. However, there seems to be no definite differences between the conventional and proposed NN in the *E* of the subgrade layer.

4. Conclusions

In this study, we proposed the alternative generation scheme of training patterns by extending the conventional noise injection learning algorithm to consider the material uncertainties for improving the robustness of the NN technique as an inverse analysis tool for the FWD test.

Firstly, was developed the wave analysis program based on the spectral element method for accurate and fast calculation in order to reduce the computing time to generate a large number of training patterns.

Secondly, was carried out the reliability analysis to investigate the effects of the modeling error that was introduced by uncertainties on material and layer thickness on the max deflection data for the differently layered pavement systems.

Finally, the effect of the modeling error was investigated in terms of inverse analysis by using the ANN technique.

It was found that the modeling errors due to uncertainties can significantly increase the inverse estimation error. And that the proposed noise injection training scheme could effectively handle the modeling error for both the bedrock boundary and the half space boundary cases.

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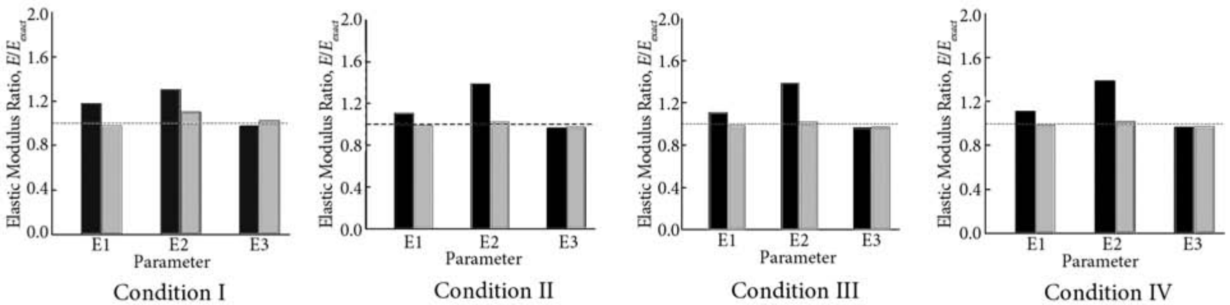

Fig. 8. Inverse analysis results for conditions with modeling errors of 5% for half space boundary case: ▬ – conventional NN; ▬ – proposed NN

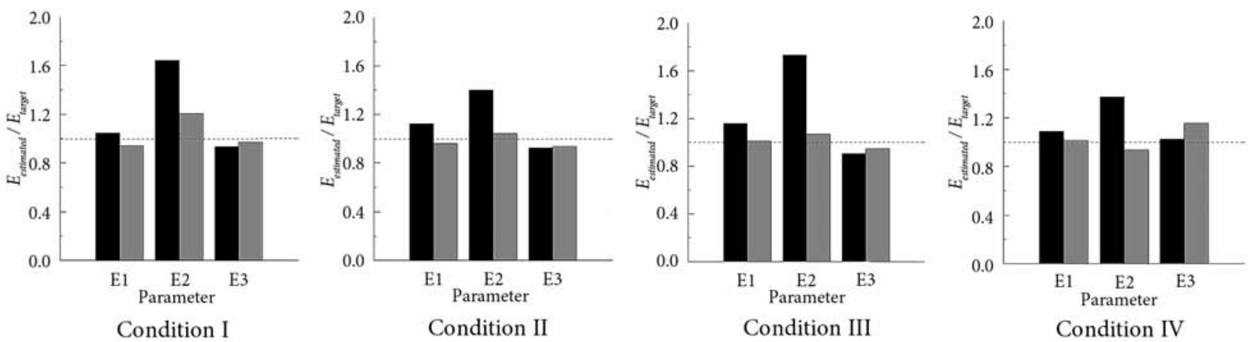

Fig. 9. Example of estimation results for conditions I, II, III, and IV with modeling errors of 5% for bedrock boundary case: ▬ – conventional NN; ▬ – proposed NN

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