



THE PHYSICALLY NONLINEAR ANALYSIS OF CIRCULAR PLATE ON DEFORMABLE FOUNDATION

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Abstract. A circular reinforced concrete plate on deformable foundation is investigated. The foundation is composed of three layers of soil which are defined as isotropically nonlinear materials with variable physical properties. The reinforced plate is treated as an elastic body. The plate is subjected to dissymmetrical half-circular loading of linearly variable pressure. Generally, the unified system “structure-foundation” is the contact analysis problem, which is modeled by ANSYS software. Plastic shear deformations are evaluated by solving the contact problem by the finite element method. The comparative analysis, including the treatment of physically linear system “structure-foundation”, is performed. The comparison based on linear analysis shows that rigidity of the foundation generates higher extreme stresses in the plate in terms of the Huber-Mises criterion. However, in nonlinear analysis, the intensity of stresses is naturally decreasing, when a more flexible model of the foundation is used.

Keywords: circular plate, deformable foundation, contact problem, physical nonlinearity, finite element method, ANSYS, plastic strains of soil, three-dimensional model.

1. Introduction

The theory of bending plates on elastic foundation employs even some foundation models (Winkler, Pasternak, Filonenko-Borodich and half space models) and is well developed and widely used in practice (Çelik, Saygın 1999; Gorbunov-Posadov *et al.* 1984; Mastrojanis 1986; Pavlik 1977; Silviera *et al.* 1999; Starovoytov *et al.* 2010; Kostantakopoulos *et al.* 2010). However, the elastic theory solutions do not properly reflect the interaction between the plate and foundation in the near edge of the plate's contact zone. More realistic and precise stress and strain distributions are derived when self-weight of a structure and layered foundation with variable physical properties are evaluated (Krutinis, Grigusevičius 2004; Regaldo *et al.* 1992). However, nonlinearly deformable foundation model will be considered if physical properties of material do not correspond to the generalized Hooke's law (Hu *et al.* 1999; De Lima *et al.* 2001; Frank, Thépot 2005; Mučinis *et al.* 2009; Olsen 1999). The relation between stresses and strains becomes nonlinear in the areas of large reactive stresses (Gorbunov-Posadov 1984; Cerioni *et al.* 1996), as well as at the edges of the foundation where shear occurs, i.e. the plastic strains emerge in the soil, as defined in the document *Shallow Foundation. Vertical Static Load Test. Preliminary Project of Experimental French Standards* (1994). It is necessary

to evaluate zones of plastic strain development below the structure, particularly for the foundations of buildings, which are sensitive to settlements.

For thorough analysis and further rational design of the plate on deformable foundation, evaluating plastic shear strains and taking into account the settlement restrictions (Skaržauskas *et al.* 2009; Tsai, Mall 2000; Yu 2006), the concept of variable repeated load and adaptability principles (Atkočiūnas 1999; Venskus *et al.* 2010) can be used. It would help to achieve more rational structural solutions, especially, for circular-like stamps and plates (Atkočiūnas *et al.* 2004; 2007), which would satisfy the requirements of limit states defined in the documents *CHиП 2.02.01-83* Основания зданий и сооружений* [Foundations of Structures and Buildings] and *CHиП 2.03.01-84* Бетонные и железобетонные конструкции* [Structures of Concrete and Reinforced Concrete] (Frank *et al.* 2004).

2. A mathematical model of the system “structure-foundation”

The full system of Eq (1) of symmetrical circular plate on elastic foundation is composed of equilibrium differential equations (*a*), geometrical differential equations (*b*), physical equations of the plate and foundation (*c, d*) and contact condition (*e*):

$$\begin{cases} [\mathcal{A}] \mathbf{M} + q(r) - p(r) = 0, & (a) \\ [\mathcal{A}]^T w(r) + \boldsymbol{\kappa} = \mathbf{0}, & (b) \\ \mathbf{M} = [K] \boldsymbol{\kappa}, & (c) \\ D(r) p(r) - s(r) = 0, & (d) \\ w(r) - s(r) = 0. & (e) \end{cases} \quad (1)$$

where $[\mathcal{A}]$ – the differential operator of equilibrium equations of the plate; $[K]$ – the stiffness matrix of the plate; \mathbf{M} – the vector of functions of the plate's bending moments; $\boldsymbol{\kappa}$ – the vector of curvature functions of the plate; $q(r)$ – a function of symmetrical external loading; $p(r)$ – a function of reactive symmetrical pressure of foundation; $w(r)$ – a function of deflections of the plate; $s(r)$ – a function of foundation settlements; $D(r)$ – a function of foundation flexibility. The described system of Eq (1) includes such unknown functions: $M_r(r)$, $M_\theta(r)$, $p(r)$, $w(r) = s(r)$, $\kappa_r(r)$, $\kappa_\theta(r)$.

When some rearrangements of equations are made, we can derive the equilibrium equation, the geometrical equation and the physical equations of the plate as follows:

$$\frac{d(rM_r)}{dr} + M_r - M_\theta = -\frac{1}{r_0} \int q r dr + \frac{1}{r_0} \int p r dr, \quad (2)$$

$$\boldsymbol{\kappa} = \begin{Bmatrix} \kappa_r \\ \kappa_\theta \end{Bmatrix} = \begin{Bmatrix} -\frac{dw}{dr^2} \\ -\frac{dw}{rdr} \end{Bmatrix}, \quad (3)$$

$$\mathbf{M} = \begin{Bmatrix} M_r(r) \\ M_\theta(r) \end{Bmatrix} = [K] \boldsymbol{\kappa} = \mathcal{K} \begin{Bmatrix} \kappa_r + v_1 \kappa_\theta \\ \kappa_\theta + v_1 \kappa_r \end{Bmatrix}, \quad (4)$$

where $\mathcal{K} = \frac{E_1 h^3}{12(1-v_1^2)}$ – the cylindrical stiffness of the plate.

3. A circular plate on deformable foundation

The circular fairly flexible plate of reinforced concrete subjected to asymmetrical loading as well as layered deformable foundation (Fig. 1) are analyzed as contact problem, taking into account the full system of Eq (1). The depth of foundation is $d = 3.0$ m, the radius of the plate is $R = 3.0$ m and the thickness $t = 0.120$ m, the elasticity modulus of concrete is $E = 30$ GPa, concrete density $\rho = 2100$ kg/m³ and Poisson's ratio $v = 0.3$. The half-circular loading area is bound by the internal $R_i = 2.0$ m, and the external $R_e = 2.5$ m radii, while the max intensity of distributed load is $p_1 = 6000$ kN/m², and $p_2 = 5000$ kN/m². The loading intensity on the soil surface is calculated depending on cutting depth and soil density by the expression $p_0 = p_{01}gd$. The elastic foundation of three layers is bound by the space $R_0 = 6.0$ m, $z_{01} = 3.0$ m, $z_{02} = 6.0$ m, $z_{03} = 11.0$ m. The physical-mechanical properties of the foundation layers are described in Table 1.

Table 1. The physical-mechanical properties of soil layers

The property of soil layer	Fine sand	Sandy loam	Clay loam
	①	②	③
Voids ratio e	0.7	0.65	0.55
Density ρ_0 , kg/m ³	1650	2050	2180
Strain modulus E'_0 , MPa	38	17	45
Poisson's ratio v_0	0.30	0.33	0.33
Cohesion of soil c , kPa	10	31	30
Angle of friction ϕ , deg	35	24	28

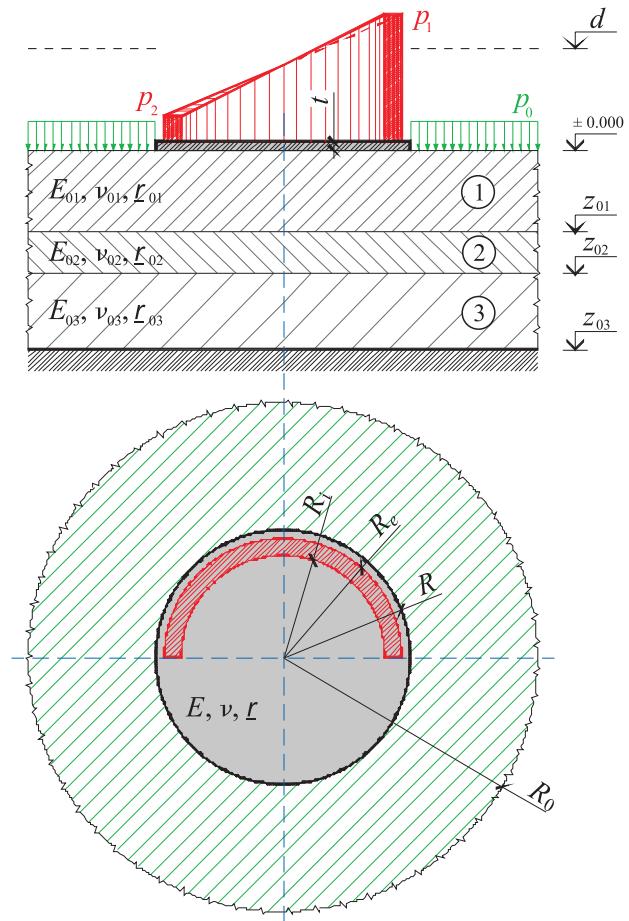


Fig. 1. Design schema of the structure

4. Physically nonlinear layered model of foundation

The dependency $\sigma-\epsilon$ of physically nonlinear material is defined for every layer under the plate at the stage of plastic shear strains development by soil test stamp (Fig. 2). The foundation yield stress R_y (adequate to the rise of plastic shear strains) and the foundation limit strength R_u (adequate to the rise of soil extrusion) are depicted on stress axis. The values of these stresses must be known for foundation design (Dirgeliene et al. 2007). However, in this paper, the main criterion is considered to be the development of soil plastic strains zones under the structure and settlement control, i.e. the stresses under the plate must

not exceed the yield stress of foundation $f_{H-M}(\sigma) \leq R_y$, and the plate settlements must not exceed the limit settlements $s \leq s_{lim}$ (Xing *et al.* 2008). Sometimes, it is advisable to allow the formation of local plastic shear strains zones, herewith achieving the more rational structural solution, which fully secures the requirements of limit states (Kala *et al.* 2009).

The foundation yield stress is calculated by the expression (Amšiejus *et al.* 2009; 2010; Šimkus 1984):

$$R_y = \frac{\pi(\gamma z_{max} + \gamma'd + c \times \operatorname{ctan}\varphi)}{\operatorname{ctan}\varphi + \varphi - \frac{\pi}{2}} + \gamma'd, \quad (5)$$

where z_{max} – max depth of soil limit strength state distribution below the foundation, m; γ – unit weight of soil below the foundation, kN/m³; γ' – unit weight of soil above the foundation, kN/m³; c – cohesion of soil below the foundation, kPa; φ – angle of friction of soil below the foundation, rad.

The convenient approximation of σ - ε diagram (Fig. 2) is performed by cubic *Bézier* curvature (Lengyel 2004), which is defined by four control points P_i , $i \in 0, 1, \dots, 3 \Leftrightarrow i \in I$, and can be expressed as follows:

$$\mathbf{B}(t) \equiv \begin{Bmatrix} \varepsilon(t) \\ \sigma(t) \end{Bmatrix} = \begin{bmatrix} P_i | i \in I \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}. \quad (6)$$

The proposed *Bézier* approximation is implemented in *MATLAB* environment (Jankovski, Atkočiūnas 2008; 2010), and is applied for every soil layer of the foundation, as well as approximates a zone of plastic shear strains by 0.25 steps with respect to boundary values of R_y and $R_{y,0.25}$ for the purpose of creating the ANSYS *Multilinear Kinematic Hardening* (MKIN) curve (Fig. 3).

Finally, the approximated values of σ - ε dependency will be used for creating physically nonlinear models of foundation layers in ANSYS software (Fig. 4). The physical model of isotropic material will be used for idealization.

5. Structural model generation in the ANSYS pre-processor

Structural model generation of the structure “plate-foundation” starts with the creation of the geometric discrete model (GDM). Indeed, it is a planar 2D profile, an axisymmetric contour of the future foundation (Fig. 5). In addition, the profile is divided into separate areas A_1-A_6 composed of keypoints K_1-K_{14} and lines L_1-L_{19} . These defined areas will be responsible for meshing control and densification, which is described by a_i-d_i parameters. The foundation model is divided by a mapped three-dimensional finite element mesh. It is advisable to model the foundation as a solid of revolution. Its profile could be divided by quadrangle elements MESH200 of desirable density and distribution, specially intended for this type of modeling.

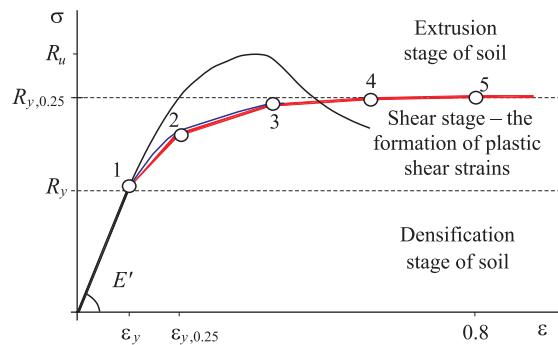


Fig. 2. The stamp test: idealized diagram of soil σ - ε dependency

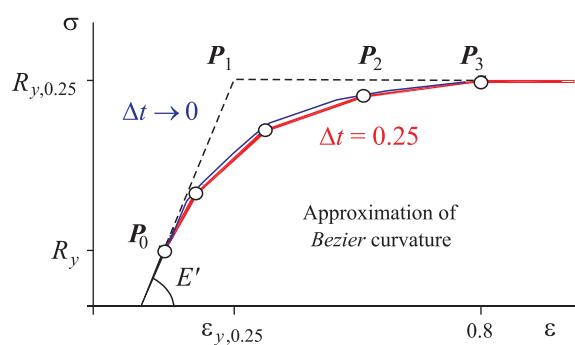


Fig. 3. Approximation of plastic shear strains zone by 0.25 steps

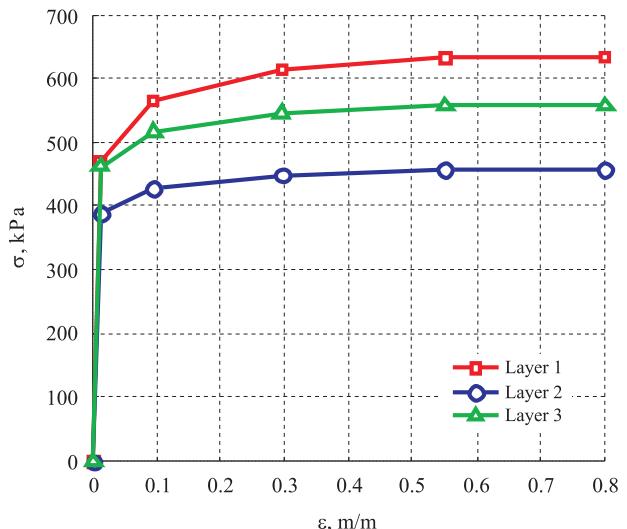


Fig. 4. Nonlinear physical models MKIN of the materials layers

The loading of asymmetrical half-circular linearly variable pressure is applied to the model of “plate-foundation” structure. The circular plate of reinforced concrete is of uniform thickness and meshed by plane quadrangle finite elements SHELL63. The plate is fairly flexible, therefore it is very likely that the contact with the foundation will be lost due to asymmetrical half-circular pressure. Therefore it is advisable to investigate such a case as a contact problem (Guenfoud *et al.* 2010), applying single-sided bonds modeled by CONTAC52 finite elements. The foundation

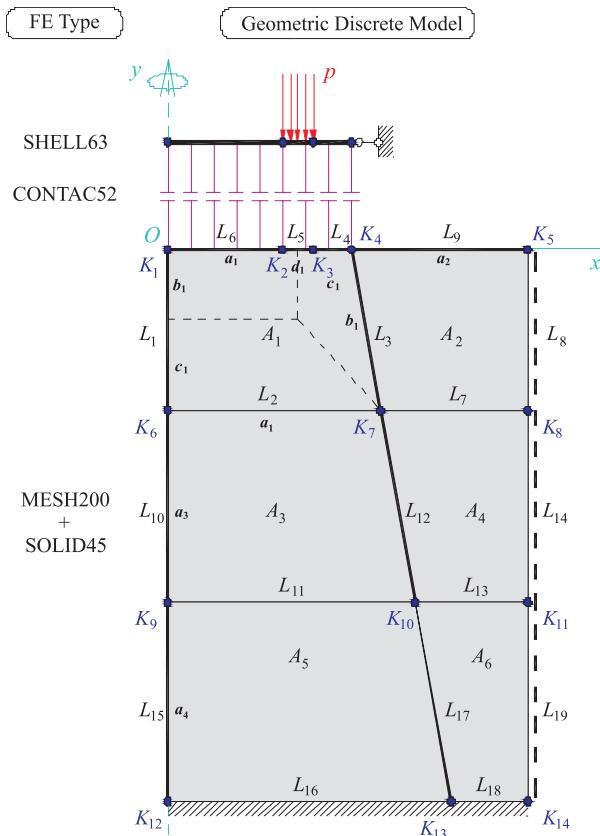


Fig. 5. The so-called “bottom up” modeling schema of circular plate on deformable foundation

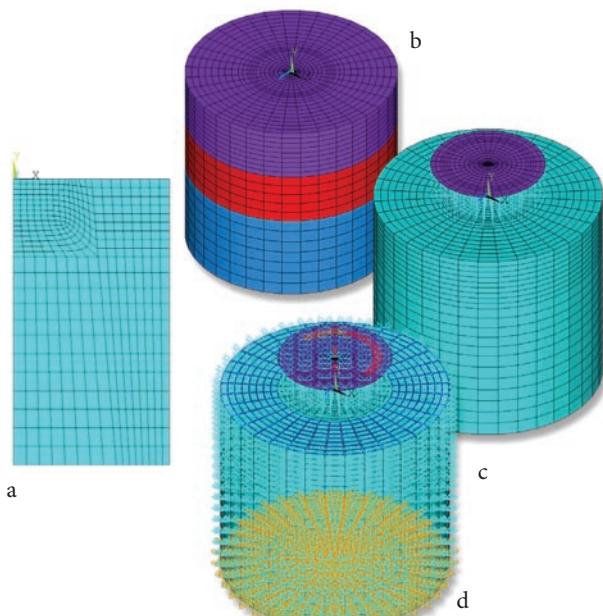


Fig. 6. Preparatory and intermediate phases of modeling: a – the planar profile discretized by the MESH200 finite elements; b – foundation of the solid of revolution; c – the system “structure-foundation”; d – the final finite element model with boundary conditions

model is composed of three soil layers of different physical-mechanical characteristics which are discussed above. These layers are modeled by 3D prismatic finite elements SOLID45 (Barauskas *et al.* 2004; Cook *et al.* 1989; Cook 1995; Gallagher 1975; Wilson 2002; Zienkiewicz 1977; Образцов *et al.* 1985; Wang *et al.* 2003).

The modeling schema (Fig. 5) is used for the fully parametric batch and the formation of the initial data ANSYS file composed of the pre-processor, solution and post-processor stages. Since the system model is fully parametrized, it allows us to modify all the initial conditions, to analyze the results and to discuss suitability of the design structure (Skaržauskas *et al.* 2006).

The following preparatory and intermediate phases were derived: nonlinear physical-mechanical models of layered foundation; the preliminary rotational-planar profile of the foundation for generating 3D solid of revolution of finite element model (Fig. 6a); the foundation of the solid of revolution discretized by the prismatic finite elements (Fig. 6b); the finite element model of the system “structure-foundation” (Fig. 6c); boundary conditions of the model’s surfaces; transfer of the boundary conditions to the finite element model (Fig. 6d).

6. The results obtained and their comparative graphical interpretation

The performed comparative linear (L) and nonlinear (NL) static analysis of the circular plate on deformable foundation allowed us to interpret visually the results by the ANSYS post-processor (Figs 7–13a).

The strains on the soil surface under shear occur and spread in the layers of soil along the plate perimeter (Fig. 14).

The variation of max settlement of the structure’s node N1011 in the time history for linear and nonlinear analysis is shown in Fig. 15.

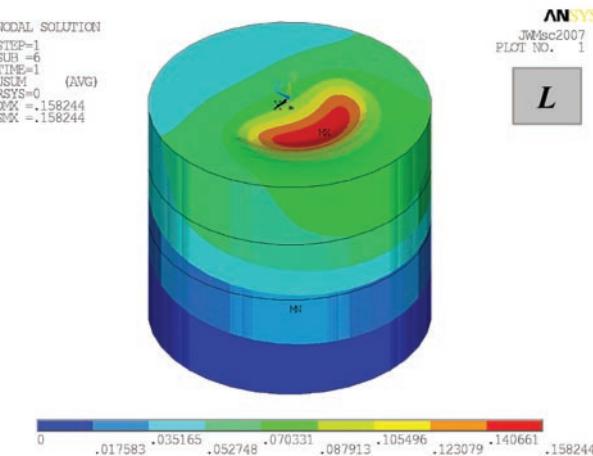
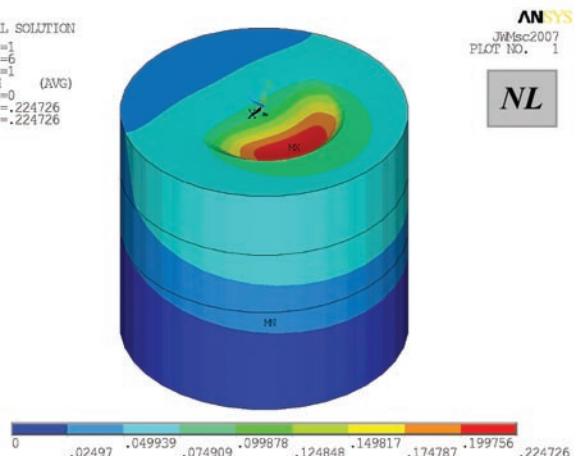
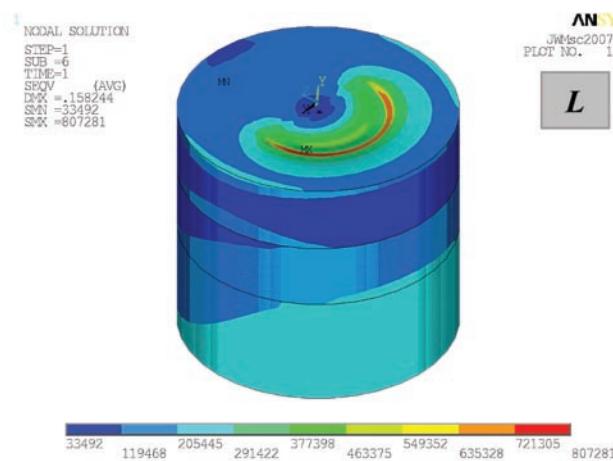
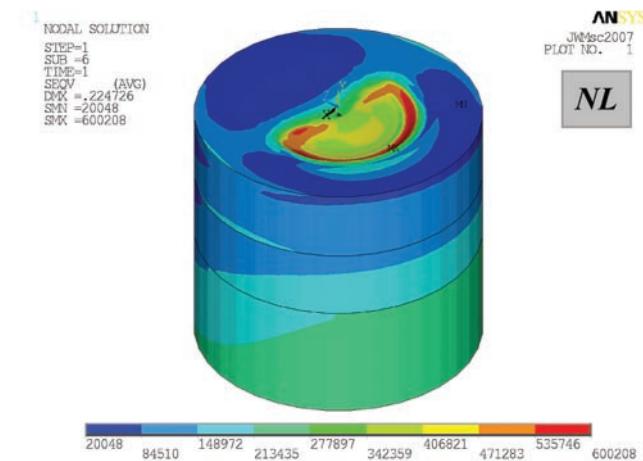
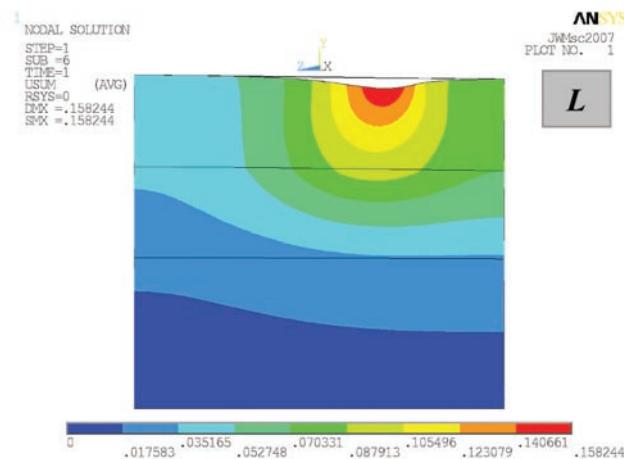
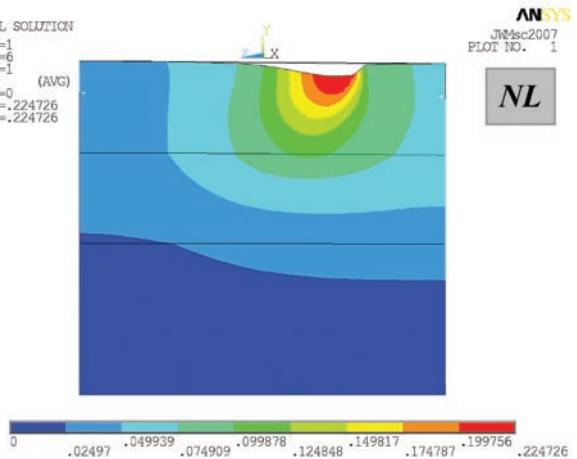
7. Conclusions

The results of the comparative analysis of the circular plate on deformable foundation show that, in the case of physical nonlinearity, the stress-strain state in the soil with plastic shear residual strains is closer to reality than that in the case of commonly used analysis of structures on deformable foundation.

The increase of the plate settlements from 15.8 cm in linear analysis to 22.5 cm in nonlinear analysis is strongly recommended for evaluating physical nonlinearity. The strains on the soil surface under shear occur and spread in the layers of soil along the plate’s perimeter.

The intensity of stresses in the foundation decreased from 0.8 MPa to 0.6 MPa in linear analysis compared to nonlinear analysis due to the development of soil strains.

Max stress intensities in the plate under the Huber-Mises criterion increased from 161 MPa to 182 MPa due to the increment of soil flexibility in nonlinear analysis.

**Fig. 7.** Foundation settlements $s_L \in [0; 0.158]$, m**Fig. 7a.** Foundation settlements $s_{NL} \in [0; 0.225]$, m**Fig. 8.** Stress intensity contours according to von Mises in the layered foundation $f_{H-M,L}(\sigma) \in [33.492; 807.281]$, kPa**Fig. 8a.** Stress intensity contours according to von Mises in the layered foundation $f_{H-M,NL}(\sigma) \in [20.048; 600.208]$, kPa**Fig. 9.** Foundation settlements in the section per node of the max settlement $s_{max,L} = 0.158$ m**Fig. 9a.** Foundation settlements in the section per node of the max settlement $s_{max,NL} = 0.225$ m

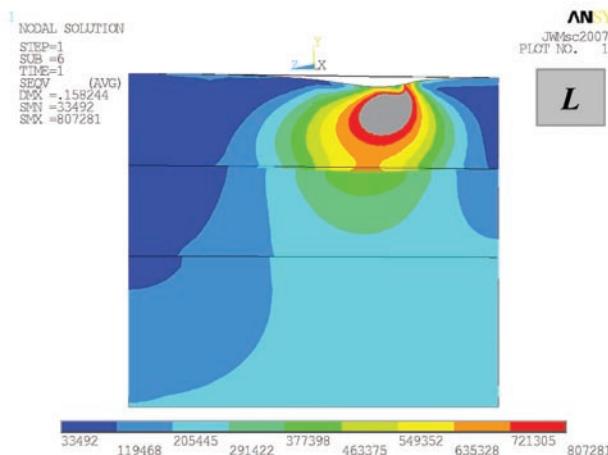


Fig. 10. Stress intensity contours according to *von Mises* in the section (linear analysis), Pa

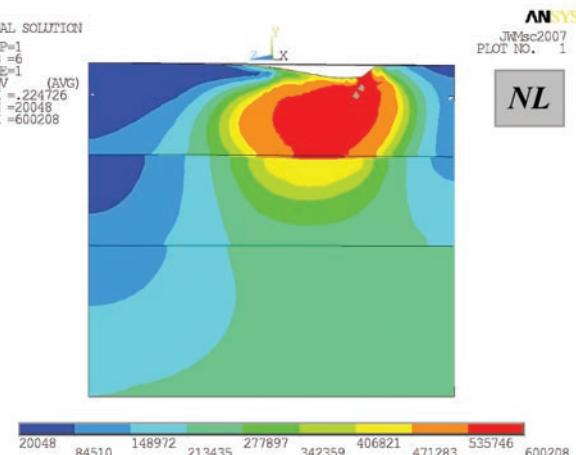


Fig. 10a. Stress intensity contours according to *von Mises* in the section (nonlinear analysis), Pa

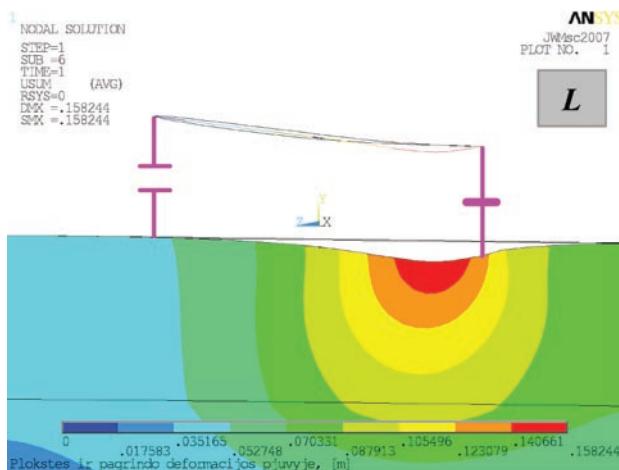


Fig. 11. Deflections of the plate and foundation in the section, m

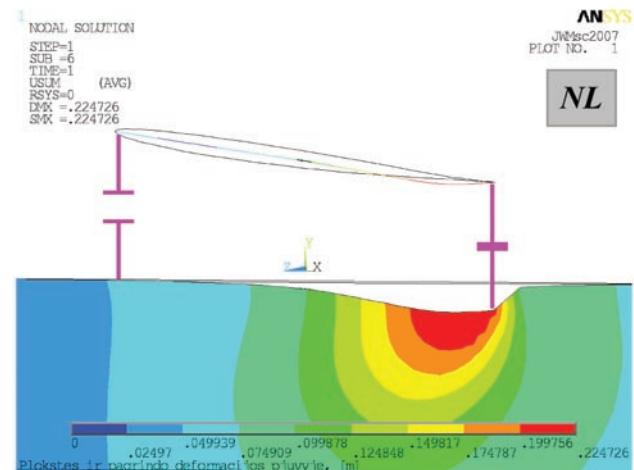


Fig. 11a. Deflections of the plate and foundation in the section, m

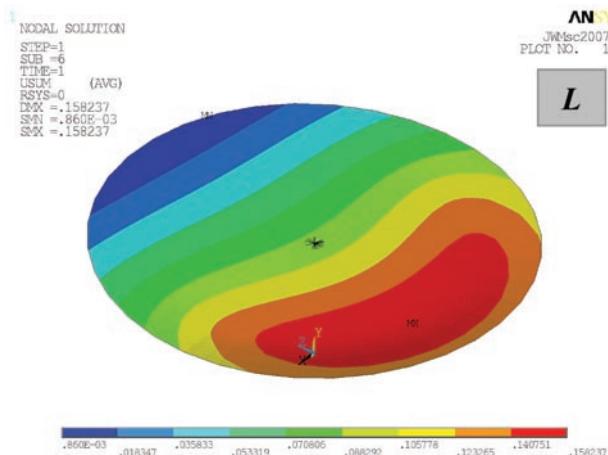


Fig. 12. Plate deflections (linear analysis), m

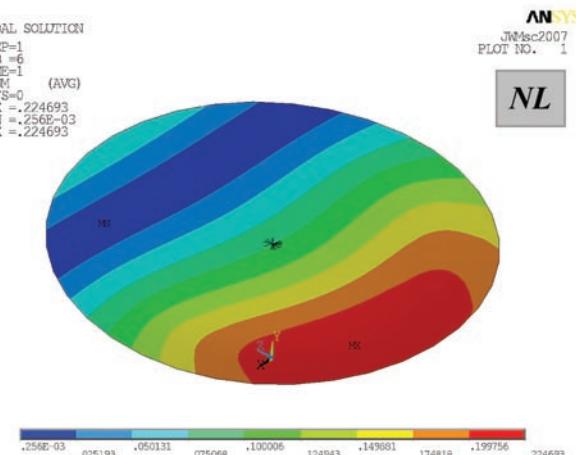


Fig. 12a. Plate deflections (nonlinear analysis), m

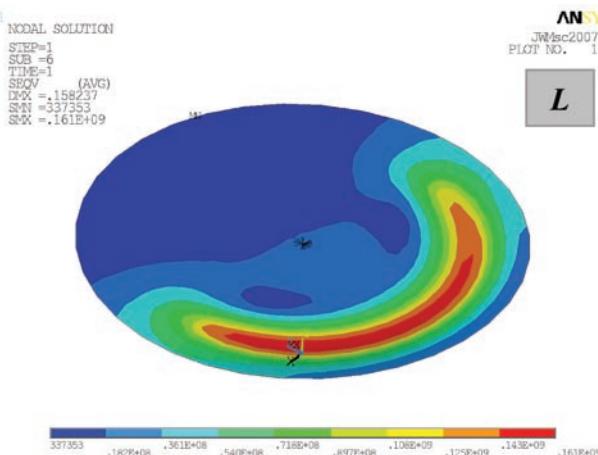


Fig. 13. Stress intensity contours according to *von Mises* in the plate $f_{H-M,L}(\sigma) \in [0.337; 161.0]$, MPa

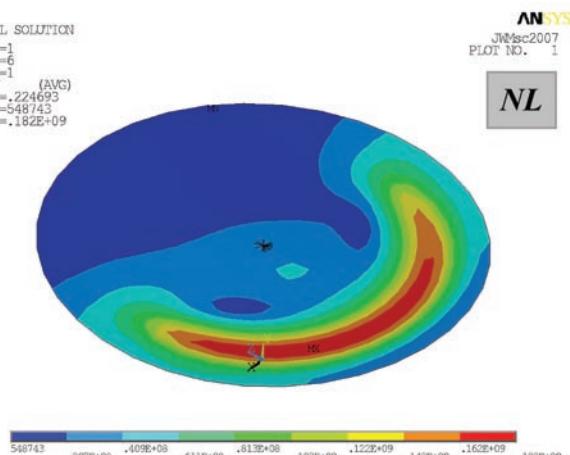


Fig. 13a. Stress intensity contours according to *von Mises* in the plate $f_{H-M,NL}(\sigma) \in [0.549; 182.0]$, MPa

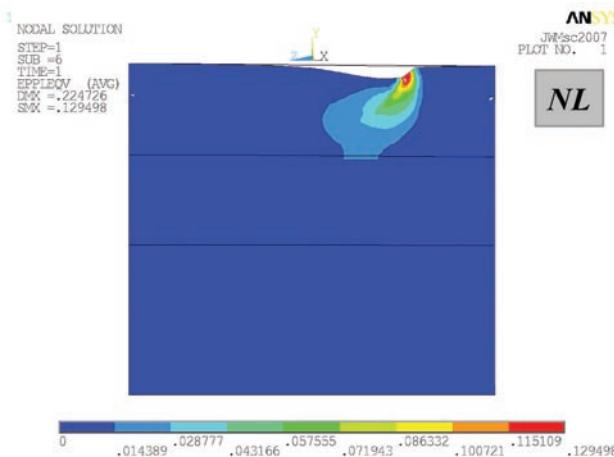


Fig. 14. Distribution of plastic shear strains according to *von Mises* criterion in the soil $f_{H-M,NL}(\epsilon) \in [0; 0.129]$, m/m

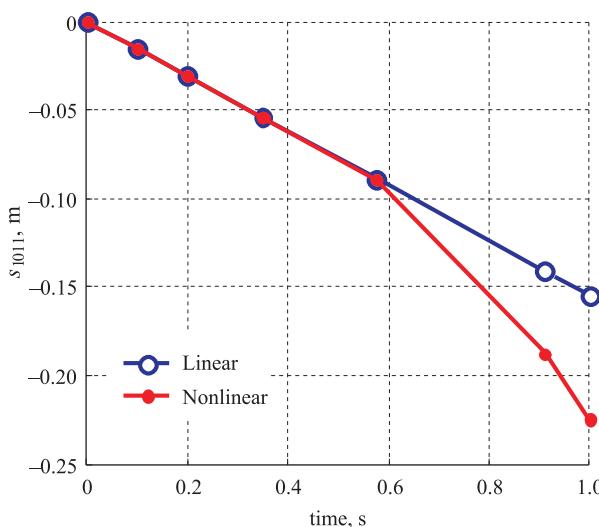


Fig. 15. Settlements $s_{1011}(time)$ of foundation node N1011 in load history with and without physical nonlinearity evaluation

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