



DISCRETE ANALYSIS OF ELASTIC CABLES

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Abstract. This paper presents a discrete calculation method for an elastic cable loaded by static concentrated forces. The discrete method is suitable to use for all suspension structures (bridges, roofs). In the calculation of the elastic cable the main problem is the geometrically non-linear behaviour of the parabolic cable. The linear methods of analysis are suitable only for small spans. A geometrically non-linear continual model is especially useful for classical loading types, e.g. uniformly distributed loads. The discrete model of suspension structures allows applying all kinds of loads, such as distributed or concentrated ones. The assumptions of the discrete method described here are: the stress-strain dependence of the material is linear, the area of the cross-section of the cable is unchangeable during the elongation and the flexural rigidity of the cable is not taken into account. An experimental investigation was conducted to prove this calculation method.

Keywords: cable-supported structure, elastic cable, suspension structure, long-span structure, discrete analysis, geometrical non-linearity, load test.

1. Introduction

A method is presented here to calculate an elastic cable using the numerical discrete analysis and the basis of the analytical discrete analysis is presented also. The geometrical non-linearity is taken into account. The supporting nodes of the cable can locate on different levels and the cable is loaded by static concentrated forces. The following assumptions were made: the stress-strain dependence of the material is linear, the cross-sectional area of the cable remains unchanged and the flexural rigidity of the cable is not taken into account. Focus in other studies may be on the utilization of cables with flexural rigidity and developing the corresponding calculation methods (Fürst *et al.* 2001; Grigorjeva *et al.* 2004, 2010a, 2010b; Juozapaitis *et al.* 2010).

Despite the fact that the calculations in the discrete method require more computational power than in the continual model, it makes it possible to apply all kinds of loads, such as distributed or concentrated ones. A geometrically non-linear continual model is especially useful for simpler loading types (e.g. a uniformly distributed load). Both of the methods give quite similar results (Aare, Kulbach 1984; Idnurm 2004; Kulbach 1999, 2007; Kulbach *et al.* 2002; Leonard 1988). The biggest problem of the discrete analysis is the huge amount of cubic and quartic equations that should be calculated. Extensive simplifications have been made in previous studies to solve this problem.

This paper presents a new algorithm to increase the accuracy of the calculation results.

Under the action of concentrated forces the cable takes the form of a string polygon. Discrete analysis is based on the equilibrium of the balanced condition composed for every nodal point of the cable. Elongation of the cable is determined using the equation of deformation compatibility for every straight section of the cable. These conditions generate a nonlinear equation system, the solution of which gives all node displacements and internal forces in the cable. The final solution (displacements and internal forces) is found by describing the initial and the final balance of the cable (before and after the loading).

2. Initial balance of the cable

The initial balance (state) of the cable is the situation before deflection. The initial balance is marked with subscript “0”. The cable is loaded with concentrated forces, i.e. the cable takes the configuration of a string polygon and the cable segments between the nodal points are as straight lines.

Let us define a cable whose neighbouring nodes are denoted by indices $i - 1$, i and $i + 1$ (Fig. 1). Let us observe the nodal point i of the cable. The nodal point is in equilibrium under the action of the internal forces of two consecutive cable segments and the external concentrated load. Then the condition of equilibrium for the initial state may be presented as (Kulbach, Öiger 1986)

$$F_{0,i} + H_0 \frac{z_{0,i+1} - z_{0,i}}{a_{0,i}} + H_0 \frac{z_{0,i-1} - z_{0,i}}{a_{0,i-1}} = 0, \quad (1)$$

where $F_{0,i}$ – initial nodal load; H_0 – initial horizontal component of the cable’s internal force; $z_{0,i-1}, z_{0,i}, z_{0,i+1}$ – initial ordinates of the cable nodes; $a_{0,i-1}, a_{0,i}$ – initial horizontal distance between the nodes.

For a cable that has supporting nodes on different levels, H_0 is calculated as (Gimsing 1997)

$$H_0 = \frac{a_{0,0} \sum_{i=1}^n F_{0,i} (L_0 - x_{0,i})}{L_0 (z_{0,1} - z_{0,0}) + a_{0,0} (z_{0,0} - z_{0,n+1})}, \quad (2)$$

where $z_{0,0}$ and $z_{0,n+1}$ – initial ordinates of the cable’s start and endpoint; $z_{0,1}$ – initial ordinate of node 1; L_0 – span of the cable; $x_{0,i}$ – initial horizontal distance between node i and the starting point of the cable; $a_{0,0}$ – initial horizontal distance between node 1 and the starting point of the cable.

3. Final balance of the cable

3.1. Exact analysis

After loading the cable with additional loads ΔF_i , the nodes have horizontal displacements u_i and vertical displacements w_i (Fig. 2). The condition of equilibrium in the final balance of the nodal point i is

$$F_i + H \frac{z_{0,i+1} + w_{i+1} - z_{0,i} - w_i}{a_{0,i} + u_{i+1} - u_i} + H \frac{z_{0,i-1} + w_{i-1} - z_{0,i} - w_i}{a_{0,i-1} + u_i - u_{i-1}} = 0, \quad (3)$$

where F_i – final nodal load ($F_i = F_{0,i} + \Delta F_i$); H – final horizontal component of the cable’s internal force; u_{i-1}, u_i, u_{i+1} – horizontal displacements of the nodes; w_{i-1}, w_i, w_{i+1} – vertical displacements of the nodes.

There are three unknown parameters in Eq (3): u, w and H that need extra equations to calculate them. It is done using the relative deformation of the cable. The relative deformation of the cable’s segment has been found by using Hooke’s law and the displacements of the cable’s nodal points. Equalizing them, the equation of deformation compatibility obtains in the form:

$$\frac{1}{E_c A_c} \left[H \sqrt{1 + \left(\frac{z_{0,i+1} + w_{i+1} - z_{0,i} - w_i}{a_{0,i} + u_{i+1} - u_i} \right)^2} - H_0 \sqrt{1 + \left(\frac{z_{0,i+1} - z_{0,i}}{a_{0,i}} \right)^2} \right] = \frac{\sqrt{(a_{0,i} + u_{i+1} - u_i)^2 + (z_{0,i+1} + w_{i+1} - z_{0,i} - w_i)^2}}{\sqrt{a_{0,i}^2 + (z_{0,i+1} - z_{0,i})^2}} - 1, \quad (4)$$

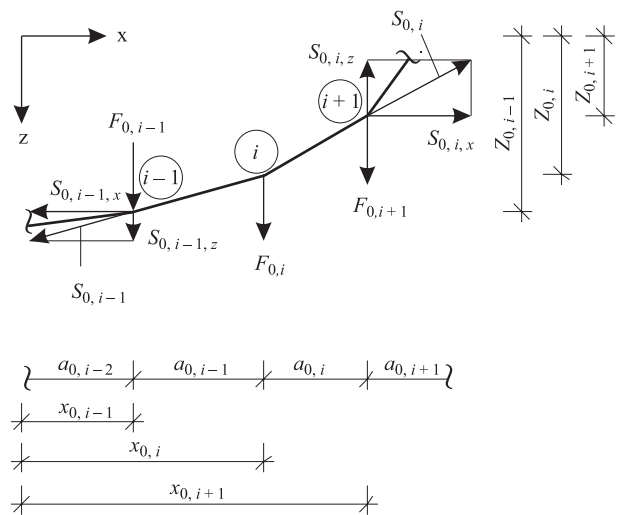


Fig. 1. Initial discrete scheme of the cable; $0 \leq i \leq n + 1$, where n – number of loads (Idnurm 2004)

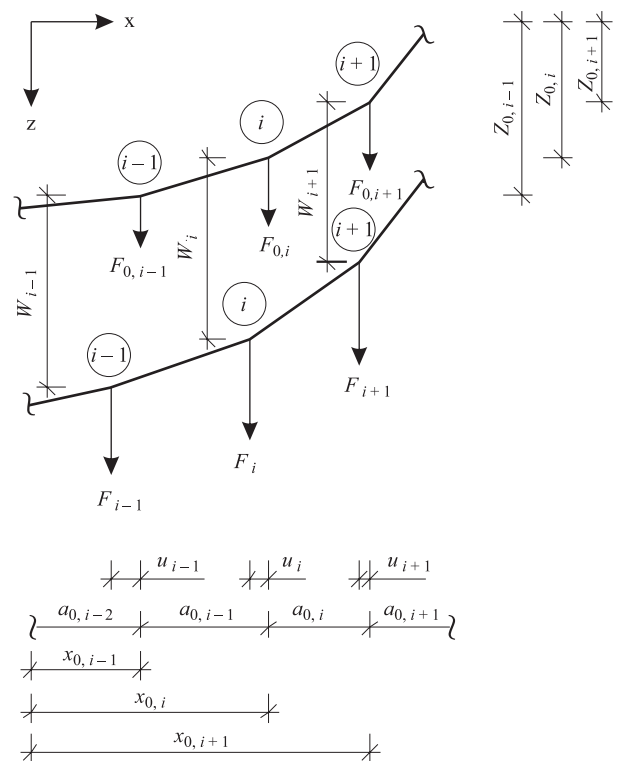


Fig. 2. Final discrete scheme of the cable (Idnurm 2004)

where E_c – cable’s modulus of elasticity; A_c – cable’s cross-sectional area.

3.2. Simplified analysis

An exact analysis in the final balance is complicated because there is a need to calculate numerous cubic and quartic equations to find $u_{i+1} - u_i$ and there is no usable analytical solution. Eqs (3) and (4) is simplified using the numerical analysis.

Provided that $(u_{i+1} - u_i) \ll a_{0,i}$, Eq (3) takes the form of (Kulbach, Öiger 1986)

$$F_i + H \frac{z_{0,i+1} + w_{i+1} - z_{0,i} - w_i}{a_{0,i}} + H \frac{z_{0,i-1} + w_{i-1} - z_{0,i} - w_i}{a_{0,i-1}} = 0. \quad (5)$$

Before Eq (4) is simplified, it is expressed as follows:

$$t^4 D_{4,j} - t^3 D_{5,j} + t^2 D_{6,j} - t D_{7,j} + D_{8,j} = 0, \quad (6)$$

where

$$\begin{aligned} t &= a_{0,i} + (u_{i+1} - u_i); \\ D_{1,j} &= (z_{0,i+1} + w_{i+1} - z_{0,i} - w_i)^2; \\ D_{2,j} &= \sqrt{a_{0,i}^2 + (z_{0,i+1} - z_{0,i})^2}; \\ D_{3,j} &= H_0 \sqrt{1 + \left(\frac{z_{0,i+1} - z_{0,i}}{a_{0,i}}\right)^2} - E_c A_c; \\ D_{4,j} &= \frac{(E_c A_c)^2}{D_{2,j}^2}; \\ D_{5,j} &= \frac{2HE_c A_c}{D_{2,j}}; \\ D_{6,j} &= H^2 + (E_c A_c)^2 \frac{D_{1,j}}{D_{2,j}^2} - D_{3,j}; \\ D_{7,j} &= \frac{2HE_c A_c D_{1,j}}{D_{2,j}}; \\ D_{8,j} &= H^2 D_{1,j}; \\ 1 \leq j &\leq n+1; \\ 1 \leq i &\leq n. \end{aligned}$$

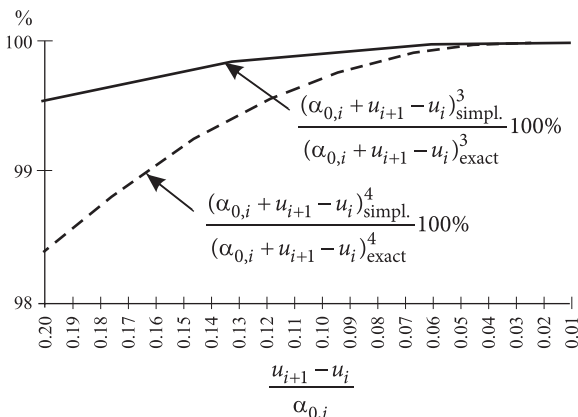


Fig. 3. Ratio of simplified results to exact results

To calculate Eq (6) it occurs that if $(u_{i+1} - u_i) \ll a_{0,i}$, the following simplifications are suitable for use (a satisfying result is attained if $\frac{u_{i+1} - u_i}{a_{0,i}} < 0.1$; Fig. 3):

$$\begin{aligned} \text{a)} \quad & \left[a_{0,i} + (u_{i+1} - u_i) \right]^3 \approx a_{0,i}^3 + 3a_{0,i}^2 (u_{i+1} - u_i) + 3a_{0,i} (u_{i+1} - u_i)^2, \\ \text{b)} \quad & \left[a_{0,i} + (u_{i+1} - u_i) \right]^4 \approx a_{0,i}^4 + 4a_{0,i}^3 (u_{i+1} - u_i) + 6a_{0,i}^2 (u_{i+1} - u_i)^2. \end{aligned}$$

This simplification leads us to the following formula:

$$u_{i+1} - u_i = D_{13,j}, \quad (7)$$

$$\begin{aligned} \text{where} \quad D_{10,j} &= 6a_{0,i}^2 D_{4,j} - 3a_{0,i} D_{5,j} + D_{6,j}; \\ D_{11,j} &= 4a_{0,i}^3 D_{4,j} - 3a_{0,i}^2 D_{5,j} + 2a_{0,i} D_{6,j} - D_{7,j}; \\ D_{12,j} &= a_{0,i}^4 D_{4,j} - a_{0,i}^3 D_{5,j} + a_{0,i}^2 D_{6,j} - a_{0,i} D_{7,j} + D_{8,j}; \\ D_{13,j} &= \frac{-D_{11,j} + \sqrt{D_{11,j}^2 - 4D_{10,j} D_{12,j}}}{2D_{10,j}}. \end{aligned}$$

Taking into account that (Kulbach, Öiger 1986)

$$\sum_{i=1}^n (u_{i+1} - u_i) = u_{n+1} - u_0, \quad (8)$$

where u_{n+1}, u_0 – horizontal displacements of the support nodes of the cable, Eq (7) may be written in the form of

$$u_{n+1} - u_0 = \sum_{j=1}^{n+1} D_{13,j}. \quad (9)$$

The final internal force of the cable and the displacements of the nodes may be calculated using a one- or a two-level iterative process (respectively steps 1...5 and 6...8 below). The solution algorithm is presented below.

1. Use F_i in Eqs (1) and (2) to calculate estimated H .
2. Use Eq (5) to calculate w_i .
3. Use Eq (9) to calculate $u_{n+1} - u_0$.
4. Compare the calculated $u_{n+1} - u_0$ to the exact value (for example – if the supports are fixed, then $u_{n+1} - u_0 = 0$). If the difference between them is not small enough, modify the value of H and repeat the calculation from step 2.
5. Use Eq (7) to calculate u_i . The first level of the iterative process is completed.
6. Here starts the second level of the iterative process. Take H and u_i from the first iterative process and use Eq (3) to calculate w_i .
7. Use Eqs (7) and (9) to calculate corrected H and u_i .
8. Repeat steps 6 and 7 until H, u_i and w_i are converged to the required precision.

If the deflections of the cable are relatively small (the experiments and the calculations showed that the vertical deflection should be less than $\frac{L}{200}$), then it is accurate enough to use the one-level iterative process (further: the first simplification). It is not recommended to calculate the horizontal displacements of the cable's nodal points (except the cable's supports) using the first simplification.

The full two-level iterative process (further: the second simplification) requires a high computational efficiency, but if $\frac{(u_{i+1} - u_i)}{a_{0,i}} < 0.1$, this method is exact in practice.

4. Numerical results

To characterize the behaviour of the cable under the concentrated loads, the numerical results were calculated. For that purpose a cable with a span of 50 m was chosen. Supports of the cable are on different levels (the vertical distance between them is 15 m). In the initial balance all 4 nodes of the cable were loaded by a concentrated load of 50 kN and in the final balance 100 kN was added. The cross-sectional area of the cable was $A_c = 2228 \text{ mm}^2$ and the modulus of elasticity $E_c = 1.25 \times 10^5 \text{ N/mm}^2$. The design scheme of the structure in the initial balance is presented in Fig. 4.

Results of the calculation (vertical and horizontal displacements of the nodal points and internal force of the cable) using the first and the second simplification of the discrete analysis are presented in Table 1 (Fig. 5). The geometrically non-linear behaviour of the cable under concentrated nodal loads between 50...150 kN is illustrated in Fig. 6.

5. Experimental investigation

The numerical calculation method presented in this paper was checked experimentally. The cable structure is the same as presented in Fig. 4. The scale of the model is

$$\alpha = \frac{1}{25} = 0.04. \text{ Parameters of the model are presented in}$$

Table 2 and some pictures of the model in Figs 7–10. The vertical displacements of the nodal points and the internal force of the cable were measured.

Table 1. Numerical results of the example

Parameter	Unit	First simpl. S_1	Second simpl. S_2	$\frac{S_1}{S_2} \cdot 100\%$
H ,	N	1 297 733	1 284 067	1.1
w_1	mm	311.7	322.8	-3.4
w_2	mm	467.6	470.6	-0.6
w_3	mm	467.6	452.8	3.3
w_4	mm	311.7	288.1	8.2
u_1	mm	55.2	55.4	-0.4
u_2	mm	115.2	113.5	1.5
u_3	mm	147.8	140.2	5.4
u_4	mm	120.4	108.5	11.0

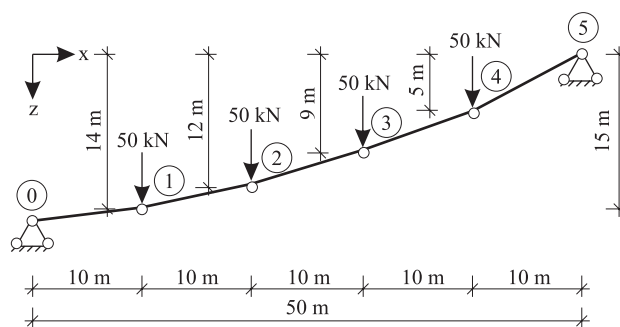


Fig. 4. Design scheme of the cable structure in the initial balance

Table 2. Parameters of the structure and the model

Parameter		Cable	Coeff.	Model
Over. dim.	Span	50 m	α	2000 mm
	Height	15 m	α	600 mm
Cable	Sect. area	2228 mm ²	α^2	3.565 mm ²
	Mod. of elasticity	$1.25 \times 10^5 \text{ MPa}$	1	$1.25 \times 10^5 \text{ MPa}$
Loads	Init. bal.	50 kN	α^2	80 N
	Final bal.	150 kN	α^2	160 N

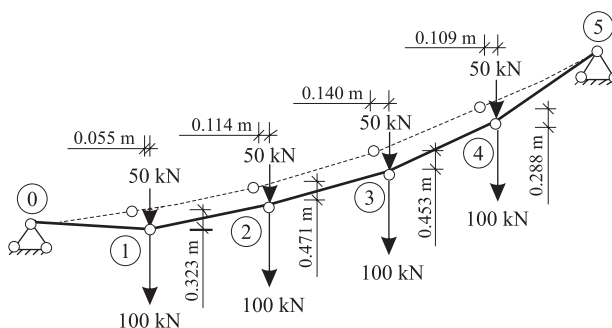


Fig. 5. Displacements of the cable's nodal points in the final balance using the second simplification (the vertical displacements are increased 5 and the horizontal displacements 10 times in the figure)

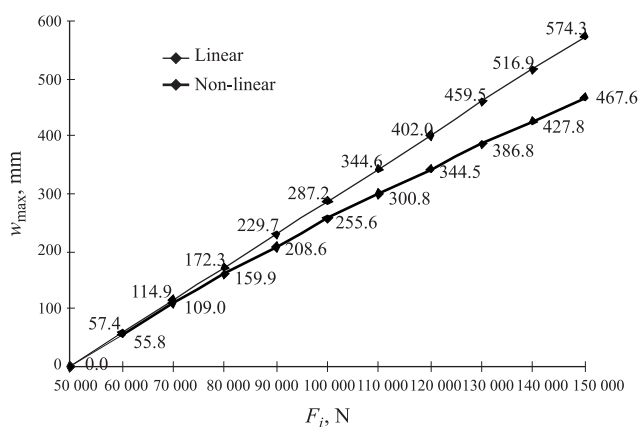


Fig. 6. Linear and non-linear relations between the nodal load F_i and the max vertical deflection of the cable w_{max} (based on the first simplification)

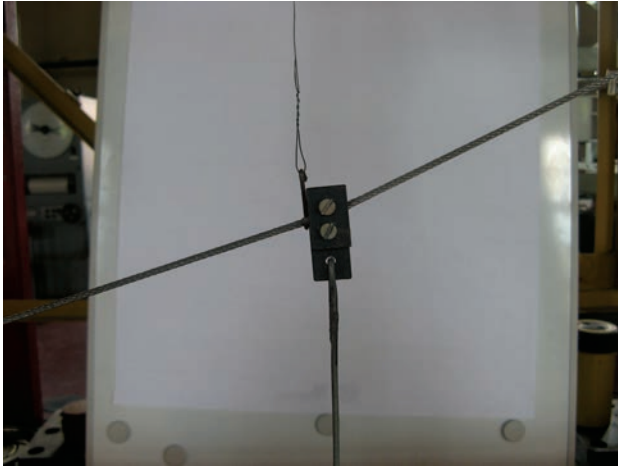


Fig. 7. Nodal point of the cable where the vertical displacements were measured

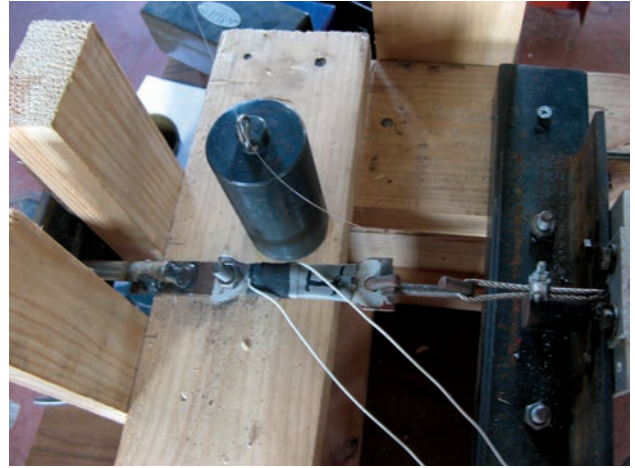


Fig. 10. Strain gauges to measure the elongation of the cables

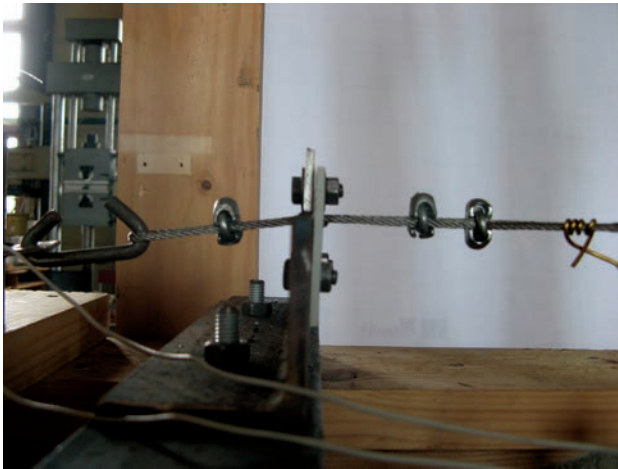


Fig. 8. Support of the cable



Fig. 9. Maksimov's gauge with the accuracy of 0.1 mm to measure the vertical displacements of the nodal points

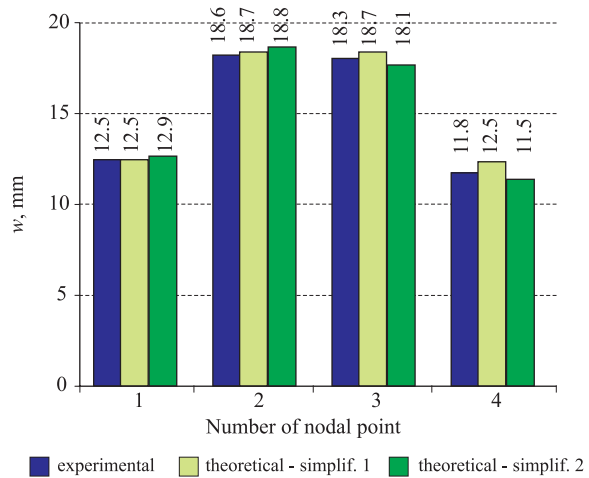


Fig. 11. Comparison of max vertical displacements of nodal points between experimental results and numerical calculations (model scale 1:25)

Comparison between the test results proved the reliability of the model. Results among each testing varied at a max of $\pm 1\%$ from the average. The average difference between single tests and mean results was 0.0%. That was the main reason why only five tests were done.

Comparison between the average experimental vertical displacement of each nodal point and the calculations was in the range of $+0.4\%$... -5.1% using the first simplification and $+2.7\%$... -3.1% using the second simplification (Fig. 11). The experimental horizontal component of the cable's internal force was 6.7% and 5.7% smaller than the calculations based on two simplified algorithms showed. Measured displacements of nodal points were on average 1.9% and 0.2% smaller (accordingly, compared to the first and the second simplification) than the calculations predicted, which proves that the numerical calculation method worked out in this paper is usable.

6. Conclusion

This article provides an algorithm to calculate internal forces and deflections of an elastic cable using the discrete analysis. Three solutions are presented – an exact analysis (analytical) and two simplified methods (numerical). An experimental investigation was also carried out to verify the simplified calculation methods.

Using the exact analysis in the final balance is complicated because it leads to non-linear equations that have no usable analytical solutions. The idea of a simplification is that some parameters that have inconsiderable influence on the final result are eliminated from the equations. As a result, all cubic and quartic equations are transformed to quadratic equations.

The consequences of the numerical calculation methods.

1. If the vertical deflection of the cable is relatively small (the experiments and the calculations showed that it should be less than $\frac{L}{200}$), it is accurate enough to use

the first simplification (one-level iterative process) to calculate the vertical deflections and the internal forces of the cable. It is not recommended to calculate the horizontal displacements of the cable's nodal points (except the cable's supports) using this method.

2. The full two-level iterative process uses simplifications only in the equations of the relative deformation of the cable's segments. If $\frac{(u_{i+1} - u_i)}{a_{0,i}} < 0.1$, this method is

exact in practice. The disadvantage of this method is that it requires high computational efficiency.

The consequences of load-testing.

1. The test results verified the reliability of the test model. The results of each test varied at a max of $\pm 1\%$ from the average. The average difference between single test results and mean results was 0.0%.

2. Measured vertical displacements of the nodal points were on average 1.9% and 0.2% smaller (accordingly, compared to the first and the second simplification) than the calculations predicted. The experimental horizontal component of the cable's internal force was 6.7% and 5.7% smaller than the calculations based on two simplified algorithms showed. This proves that the numerical calculation method worked out in this paper is usable.

The geometrically nonlinear numerical discrete analysis presented in this paper enables adequate determination of deflections and internal forces of the elastic cable. The numerical example demonstrated a very good agreement between the results of both the simplified discrete methods and the experimental investigation. The biggest

advantage of the discrete analysis is that it is easy to describe different load types and load combinations. The most important disadvantage is the necessity to calculate complicated systems of equations and very often these systems converge slowly. Because the accuracy of the calculations is high and the development of the digital computers is fast, the discrete analysis has a significant role in the calculations of the long-span cable-supported structures.

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